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Department of Mathematics

Lecture Notes

Calculus & Linear Algebra (18MAT11)

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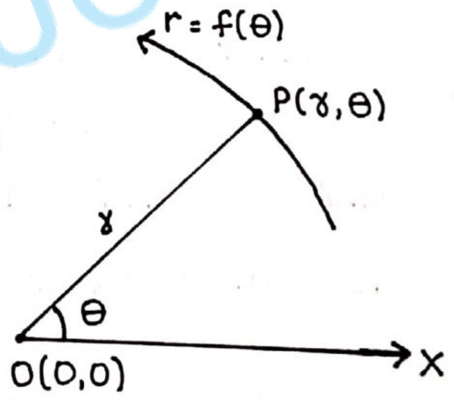
CALCULUS AND LINEAR ALGEBRA

MODULE - 01

POLAR CURVES AND EVALUATES

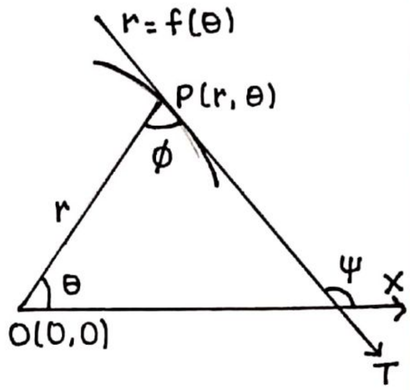
Introduction :-

Let $\overrightarrow{OX}$ be the intial line or intial ray, there is any point on the plane $P(r,\theta)$ with the radius of vector $OP = r$ and $\angle XOP = \theta$ and $r = f(\theta)$ be a polar curve at the point $P(r,\theta)$ Here, the Coordinates of P is called as the Polar Coordinates.



A. ANGLE BETWEEN RADIUS VECTOR AND TANGENT TO THE POLAR CURVE

Let $\overrightarrow{OX}$ be the intial line $P(r,\theta)$ be a point in the plane and $\overrightarrow{OP} = r$ be the radius of vector and Let $r = f(\theta)$ be a polar curve at the point 'P', 'T' be the tangent to the curve $r = f(\theta)$ at P and making angle with the intial line ψ



let ϕ be the angle between radius of vector and tangent to the curve $r = f(\theta)$ at P

W.K.T

$$\psi = \phi + \theta$$

$$\Rightarrow \tan\psi = \tan(\phi + \theta)$$

$$\tan\psi = \frac{\tan\phi + \tan\theta}{1 - \tan\phi \tan\theta} \longrightarrow (1)$$

also W.K.T The slope of the tangent T is $m = \tan\psi = \frac{dy}{dx} \longrightarrow (2)$

let $x = r\cos\theta$ and $y = r\sin\theta$

differentiate x and y w r t θ

$$\therefore \frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta$$

$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

$$(2) \Rightarrow \tan\psi = \frac{dy}{dx} \quad \frac{dy/d\theta}{dx/d\theta}$$

$$\Rightarrow \tan\psi = \frac{dr/d\theta \, \sin\theta + r\cos\theta}{dr/d\theta \, \cos\theta - r\sin\theta}$$

$$= \frac{\dfrac{\frac{dr}{d\theta}\sin\theta}{\frac{dr}{d\theta}\cos\theta} + \dfrac{r\cos\theta}{\frac{dr}{d\theta}\cos\theta}}{1 - \dfrac{r\sin\theta}{\frac{dr}{d\theta}\cos\theta}}$$

$$\Rightarrow \tan\psi = \tan\theta + r\frac{d\theta}{dr} \Big/ 1 - r\frac{d\theta}{dr}\tan\theta$$

$$\frac{\tan\theta + \tan\phi}{1 - \tan\phi\tan\theta} = \frac{\tan\theta + r\frac{d\theta}{dr}}{1 - r\frac{d\theta}{dr}\tan\theta}$$

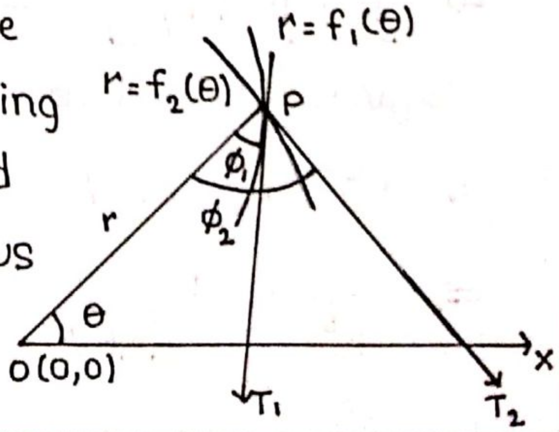
$$\boxed{\tan\phi = r\frac{d\theta}{dr}}$$

$$\phi = \tan^{-1} r\frac{d\theta}{dr}$$

Interms of, $\cot\phi = \frac{1}{r}\frac{dr}{d\theta}$

B. ANGLE BETWEEN TWO POLAR CURVES

Let $r = f_1(\theta)$, $r = f_2(\theta)$ be the given two polor curves having the tangents T_1 and T_2 and making an angles of radius of vector ϕ_1 and ϕ_2 then the angle between the

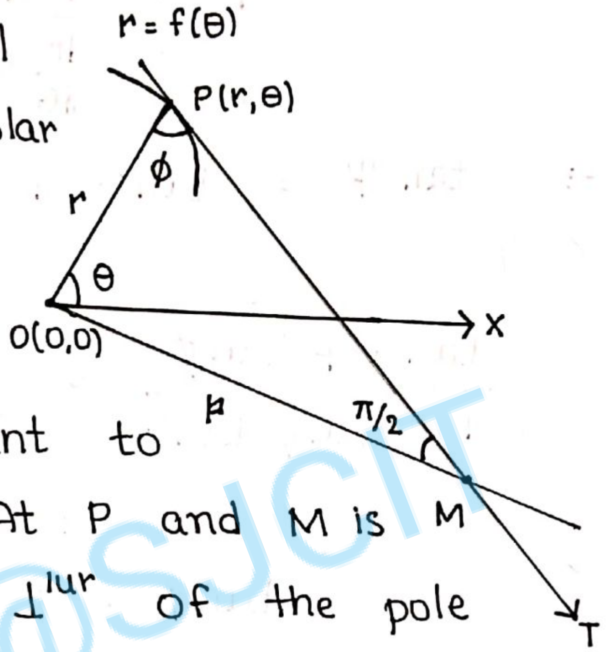


given two polar curves is the angle between there two tangent $= |\phi_2 - \phi_1|$

WITH USUAL NOTATION, PROVE THAT

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

Let $\overrightarrow{OX}$ be the intial line $r = f(\theta)$ be a polar curve at $P(r,\theta)$ and $OP = r$ be the radius of Vector.



Let T be the tangent to the curve $r = f(\theta)$. At P and M is the foot of the $\perp^{lur}$ of the pole $O(0,0)$ having the perpendicular distance $OM = p$ from the ΔOPM we have $\angle OMP = 90°$

$\therefore \sin\phi = \frac{OM}{OP}$

$\Rightarrow \sin\phi = \frac{p}{r}$

$\Rightarrow \boxed{p = r\sin\phi} \longrightarrow (1)$

Square on both Sides

$\Rightarrow p^2 = r^2 \sin^2\phi$

$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2\sin^2\phi}$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} \operatorname{cosec}^2 \phi$$

$$\boxed{\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}(1 + \cot^2 \phi)}$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left[1 + \left(\frac{1}{r}\left(\frac{dr}{d\theta}\right)\right)^2\right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left[1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right]$$

$$\boxed{\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2}$$

I. Find the angle between the radius vector and tangent for the following polar curves.

① $r = a(1 - \cos\theta)$

Given,

$r = a(1 - \cos\theta) \longrightarrow ①$

differentiate ① w.r.t. θ

$$\frac{dr}{d\theta} = a(0 + \sin\theta)$$

$$\frac{dr}{d\theta} = a \sin\theta$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{a \sin\theta}{r}$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\cancel{a} \sin\theta}{\cancel{a}(1 - \cos\theta)}$$

$$\Rightarrow \cot\phi = \frac{\sin\theta}{(1 - \cos\theta)}$$

$$\Rightarrow \cot\phi = \frac{2\sin\theta/2 \cos\theta/2}{2\sin^2\theta/2}$$

$$\Rightarrow \cot\phi = \cot\theta/2$$

$$\boxed{\phi = \theta/2}$$

When, $\theta = \frac{\pi}{3}$

$$\phi = \frac{\pi/3}{2}$$

$$\phi = \frac{\pi}{6} = 30^\circ //$$

(2) $r = a(1+\cos\theta)$

Given

$r = a(1+\cos\theta) \longrightarrow (1)$

differentiate (1) w.r.t. θ

$$\frac{dr}{d\theta} = a(0 - \sin\theta)$$

$$\frac{dr}{d\theta} = a(-\sin\theta)$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-a\sin\theta}{r}$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-a\sin\theta}{a(1+\cos\theta)}$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{(1+\cos\theta)}$$

$$\frac{1}{r}\frac{dr}{d\theta} = -\frac{2\sin\theta/2 \cos\theta/2}{2\cos^2\theta/2}$$

$$\Rightarrow \cot\phi = -\tan\theta/2$$

$$\Rightarrow \cot\phi = \cot(\pi/2 + \theta/2)$$

$$\phi = \frac{\pi}{2} + \frac{\theta}{2} \quad //$$

③ $r = a(1 + \sin\theta)$

Given

$r = a(1 + \sin\theta) \longrightarrow ①$

differentiate ① w.r.t. θ

$$① \Rightarrow \frac{dr}{d\theta} = a(0 + \cos\theta)$$

$$\Rightarrow \frac{dr}{d\theta} = a\cos\theta$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{a\cos\theta}{r}$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{a\cos\theta}{a(1+\sin\theta)}$$

$$\Rightarrow \cot\phi = \frac{\cos\theta}{1+\sin\theta}$$

$$\Rightarrow \cot\phi = \frac{\cos^2\theta/2 - \sin^2\theta/2}{\cos^2\theta/2 + \sin^2\theta/2 + 2\sin\theta/2\cos\theta/2}$$

$$= \frac{(\cos\theta/2 + \sin\theta/2)(\cos\theta/2 - \sin\theta/2)}{(\cos\theta/2 + \sin\theta/2)^2}$$

$$= \frac{\cos\theta/2 - \sin\theta/2}{\cos\theta/2 + \sin\theta/2}$$

$$= \frac{1 - \tan\theta/2}{1 + \sin\theta/2 / \cos\theta/2}$$

$$= \frac{1 - \tan\theta/2}{1 + \tan\theta/2}$$

$$= \frac{\tan(\pi/4) - \tan(\theta/2)}{1 + \tan(\pi/4)\tan(\theta/2)}$$

$$\Rightarrow \cot\phi = \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$$

$$\Rightarrow \cot\phi = \cot\left[\frac{\pi}{2} - \left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right]$$

$$\Rightarrow \phi = \frac{\pi}{2} - \frac{\pi}{4} + \frac{\theta}{2}$$

$$\Rightarrow \phi = \frac{\pi}{4} + \frac{\theta}{2} \quad //$$

II. Show that the following pairs of curves intersect each other orthogonally.

①. $r = a(1+\cos\theta)$, $r = b(1-\cos\theta)$

Given

$r = a(1+\cos\theta) \longrightarrow$ ①

differentiate ① w.r.t. θ

$$\frac{dr}{d\theta} = -a\sin\theta$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-a\sin\theta}{r}$$

$$\Rightarrow \cot\phi_1 = \frac{-a\sin\theta}{a(1+\cos\theta}$$

$$\Rightarrow \cot\phi_1 = \frac{-\sin\theta}{1+\cos\theta}$$

$$\Rightarrow \cot\phi_1 = \frac{-2\sin\theta/2\cos\theta/2}{2\cos^2\theta/2}$$

$$\Rightarrow \cot\phi_1 = -\tan\theta/2$$

$$\cot\phi = \cot\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$$

$$\therefore \phi_1 = \frac{\pi}{2}+\frac{\theta}{2}$$

Similarlly differentiate ② w.r.t. θ

$$\frac{dr}{d\theta} = b\sin\theta$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{b\sin\theta}{r}$$

$$\Rightarrow \cot\phi_2 = \frac{b\sin\theta}{b(1-\cos\theta)}$$

$$\Rightarrow \cot\phi_2 = \frac{\sin\theta}{1-\cos\theta}$$

$$\Rightarrow \cot\phi_2 = \frac{2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2}$$

$$\therefore \cot\phi_2 = \theta/2$$

$$\therefore |\phi_1-\phi_2| = \left|\frac{\pi}{2}+\frac{\theta}{2}-\frac{\theta}{2}\right| = \frac{\pi}{2} //$$

$\therefore$ the Given Curves are intersecting orthogo-nally.

② $r = a(1+\sin\theta)$, $r = a(1-\sin\theta)$

Given

$r = a(1+\sin\theta) \longrightarrow ①$

differentiate ① w.r.t. θ

$$\frac{dr}{d\theta} = a\cos\theta$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{a\cos\theta}{r}$$

$$\Rightarrow \cot\phi_1 = \frac{a\cos\theta}{a(1+\sin\theta)}$$

$$\Rightarrow \cot\phi_1 = \frac{\cos\theta}{(1+\sin\theta)}$$

$$\Rightarrow \cot\phi_1 = \frac{\cos^2\theta/2 - \sin^2\theta/2}{\cos^2\theta/2 + \sin^2\theta/2 + 2\sin\theta/2\cos\theta/2}$$

$$= \frac{\cos\theta/2 - \sin\theta/2}{\cos\theta/2 + \sin\theta/2}$$

$$= \frac{1-\tan\theta/2}{1+\tan\theta/2}$$

$$\therefore \cot\phi_1 = \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$$

$$\Rightarrow \cot\phi_1 = \cot\left[\frac{\pi}{2} - \left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right]$$

$$\phi_1 = \frac{\pi}{4} + \frac{\theta}{2}$$

Similarlly differentiate ② w.r.t. θ

$$② \Rightarrow \frac{dr}{d\theta} = -a\cos\theta$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{-a\cos\theta}{a(1-\sin\theta)}$$

$$\Rightarrow \cot\phi_2 = -\left[\frac{\cos^2\theta/2 - \sin^2\theta/2}{(\cos\theta/2 - \sin\theta/2)^2}\right]$$

$$\Rightarrow \cot\phi_2 = -\left[\frac{\cos\theta/2 + \sin\theta/2}{\cos\theta/2 - \sin\theta/2}\right]$$

$$= -\left[\frac{1+\tan\theta/2}{1-\tan\theta/2}\right]$$

$$= -\tan\left[\pi/4 + \theta/2\right]$$

$$\cot\phi_2 = \cot\left[\frac{\pi}{2} + \left(\frac{\pi}{4} + \frac{\theta}{2}\right)\right]$$

$$\therefore \phi_2 = \frac{\pi}{2} + \frac{\pi}{4} + \frac{\theta}{2}$$

$$\therefore |\phi_1 - \phi_2| = \left|\frac{\pi}{4} + \frac{\theta}{2} - \frac{\pi}{2} - \frac{\pi}{4} - \frac{\theta}{2}\right| = \frac{\pi}{2} //$$

$\therefore$ The Given curves are intersecting orthogonally

(3) $r^n = a^n(\cos n\theta)$, $r^n = b^n(\sin n\theta)$

Given

$r^n = a^n(\cos n\theta) \longrightarrow (1)$

differentiate (1) w r t θ

$$\Rightarrow n r^{n-1}\frac{dr}{d\theta} = -a^n(\sin n\theta)$$

$$\Rightarrow \frac{r^n}{r}\frac{dr}{d\theta} = -a^n\sin n\theta$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{-a^n\sin n\theta}{r^n}$$

$$\cot\phi_1 = \frac{-a^n \sin n\theta}{a^n \cos n\theta}$$

$$\cot\phi_1 = -\tan\theta$$

$$\cot\phi_1 = \cot\left(\frac{\pi}{2} + n\theta\right)$$

$$\phi_1 = \frac{\pi}{2} + n\theta$$

Similarlly differentiate ② w.r.t θ

② $\Rightarrow n r^{n-1} \frac{dr}{d\theta} = b^n n\cos\theta$

$$\Rightarrow \frac{r^n}{r} \frac{dr}{d\theta} = b^n \cos n\theta$$

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{b^n \cos n\theta}{r^n}$$

$$\Rightarrow \cot\phi_2 = \frac{b^n \cos n\theta}{r^n}$$

$$\cot\phi_2 = \cot n\theta$$

$$\phi_2 = n\theta$$

$\therefore |\phi_1 - \phi_2| = \left|\frac{\pi}{2} + n\theta - n\theta\right| = \pi/2$ //

$\therefore$ the Given curves are intesecting orthogonally

III. <u>Find the angle between the Given pairs of Curves.</u>

① $r = \sin\theta + \cos\theta$ and $r = 2\sin\theta$

Given :-

$r = \sin\theta + \cos\theta \longrightarrow$ ①

$r = 2\sin\theta \longrightarrow$ (2)

differentiate equation (1) w. r. t. θ

(1) $\Rightarrow \frac{dr}{d\theta} = \cos\theta - \sin\theta$

$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{\cos\theta - \sin\theta}{r}$

$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{\cos\theta - \sin\theta}{\sin\theta + \cos\theta}$

$= \frac{1 - \tan\theta}{1 + \tan\theta}$

$= \frac{\tan(\pi/4) - \tan\theta}{\tan(\pi/4) + \tan\theta}$

$\Rightarrow \cot\phi_1 = \tan(\pi/4 - \theta)$

$\Rightarrow \cot\phi_1 = \cot\left[\pi/2 - \left[\pi/4 - \theta\right]\right]$

$\Rightarrow \phi_1 = \frac{\pi}{2} - \frac{\pi}{4} + \theta$

III'y (2) $\Rightarrow \frac{dr}{d\theta} = 2\cos\theta$

$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{2\cos\theta}{r}$

$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{2\cos\theta}{2\sin\theta}$

$\Rightarrow \cot\phi_2 = \cot\theta$

$\Rightarrow \phi_2 = \theta$

$$\therefore \left|\phi_1 - \phi_2\right| = \left|\frac{\pi}{2} - \frac{\pi}{4} + \theta - \theta\right|$$

$$\Rightarrow \left|\phi_1 - \phi_2\right| = \pi/4 \;//$$

$\therefore$ The Given pairs of Curves are intersecting at $\frac{\pi}{4}$ $[45^\circ]$

② $r = a \log\theta$ and $r = \frac{a}{\log\theta}$

Given

$$r = a\log\theta \longrightarrow ①$$

$$r = \frac{a}{\log\theta} \longrightarrow ②$$

differentiate equation ① and ② w.r.t. θ

$$① \Rightarrow \frac{dr}{d\theta} = a \cdot \left(\frac{1}{\theta}\right)$$

$$\Rightarrow \frac{dr}{d\theta} = \frac{a}{\theta}$$

$$\Rightarrow \frac{d\theta}{dr} = \frac{\theta}{a}$$

$$\Rightarrow r\frac{d\theta}{dr} = r\frac{\theta}{a}$$

$$\Rightarrow \tan\phi_1 = \frac{(a\log\theta)\theta}{a}$$

$$\Rightarrow \tan\phi_1 = \theta\log\theta \longrightarrow ③$$

III'ly differentiate equation ② w.r.t θ

$$② \Rightarrow \frac{dr}{d\theta}\log\theta + \frac{r}{\theta} = 0$$

$$\Rightarrow \log\theta \frac{dr}{d\theta} = -\frac{r}{\theta} = 0$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = -\frac{1}{\theta\log\theta}$$

$$\Rightarrow r\frac{d\theta}{dr} = -\theta\log\theta$$

$$\Rightarrow \tan\phi_2 = -\theta\log\theta \longrightarrow (4)$$

W.K.T

$$\tan(\phi_1-\phi_2) = \frac{\tan\phi_1 - \tan\phi_2}{1+\tan\phi_1\tan\phi_2}$$

$$\Rightarrow \tan(\phi_1-\phi_2) = \frac{\theta\log\theta + \theta\log\theta}{1+(\theta\log\theta)(-\theta\log\theta)}$$

$$\Rightarrow \tan(\phi_1-\phi_2) = \frac{2\theta\log\theta}{1-\theta^2(\log\theta)^2} \longrightarrow (5)$$

$\therefore$ from equation (1) and (2)

$$a\log\theta = \frac{a}{\log\theta}$$

$$\Rightarrow (\log\theta)^2 = 1$$

$$\Rightarrow \log_e\theta = 1$$

$$\Rightarrow \theta = e^1 = e$$

$$\therefore (5) \Rightarrow \tan(\phi_1-\phi_2) = \frac{2e}{1-e^2}$$

$$\therefore |\phi_1-\phi_2| = \tan^{-1}\left(\frac{2e}{1-e^2}\right) = 2\tan^{-1}e \,//$$

$\therefore$ The Given two Curves or pair of curves are intersecting at $2\tan^{-1} e$.

③ $r^2 \sin 2\theta = 4$ and $r^2 = 16 \sin 2\theta$

Given

$r^2 \sin 2\theta = 4 \longrightarrow ① \longleftarrow \quad r^2 = \dfrac{4}{\sin 2\theta}$

$r^2 = 16 \sin 2\theta \longrightarrow ②$

$\therefore$ from equation ① and ②

$$\frac{4}{\sin 2\theta} = 16 \sin 2\theta$$

$\Rightarrow 4 \sin^2 2\theta = 1$

$\Rightarrow \sin^2 2\theta = \dfrac{1}{4}$

$\Rightarrow \sin 2\theta = \dfrac{1}{4}$

$\Rightarrow \sin 2\theta = \dfrac{1}{2}$

$\Rightarrow 2\theta = \sin^{-1}(1/2)$

$\Rightarrow 2\theta = \dfrac{\pi}{6}$

$\Rightarrow \theta = \dfrac{\pi}{12}$

differentiate equation ① and ② w.r.t.θ

① $\Rightarrow 2r \dfrac{dr}{d\theta} \sin 2\theta + r^2 \cos(2\theta) \cdot 2 = 0$

$\Rightarrow r\dfrac{dr}{d\theta} \cdot \sin 2\theta + r^2 \cos 2\theta = 0$

$\Rightarrow r\dfrac{dr}{d\theta} \cdot \sin 2\theta = -r^2 \cos 2\theta$

$$\Rightarrow r\frac{dr}{d\theta} = \frac{-r^2\cos 2\theta}{\sin\theta}$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = -\cot 2\theta$$

$$\Rightarrow \cot\phi_1 = \cot(-2\theta)$$

$$\Rightarrow \phi_1 = -2\theta$$

(2) $\Rightarrow 2r \cdot \frac{dr}{d\theta} = 16(\cos 2\theta)(2)$

$$\Rightarrow r\frac{dr}{d\theta} = 16\cos 2\theta$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{16\cos 2\theta}{r^2}$$

$$\Rightarrow \cot\phi_2 = \frac{16\cos 2\theta}{16\sin 2\theta}$$

$$\Rightarrow \cot\phi_2 = \cot 2\theta$$

$$\Rightarrow \phi_2 = 2\theta$$

$$\therefore |\phi_1 - \phi_2| = |-2\theta - 2\theta| = 4\theta$$

$$\therefore |\phi_1 - \phi_2| = 4\left(\frac{\pi}{12}\right) = \frac{\pi}{3} \quad //$$

$\therefore$ The Give pair of curves are intersecting at $\frac{\pi}{3}$

(4) $r = a(1-\cos\theta)$ and $r = 2a\cos\theta$

Given,

$$r = a(1-\cos\theta) \longrightarrow (1)$$

$r = 2a\cos\theta \longrightarrow (2)$

differentiate equation (1) and (2) w.r.t. θ

(1) $\Rightarrow \frac{dr}{d\theta} = a(0+\sin\theta)$

$\Rightarrow \frac{dr}{d\theta} = a\sin\theta$

$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{a\sin\theta}{r}$

$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{a\sin\theta}{a(1-\cos\theta)}$

$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{(1-\cos\theta)}$

$\Rightarrow \cot\phi_1 = \frac{2\sin\theta/2\,\cos\theta/2}{2\sin^2\theta/2}$

$\Rightarrow \cot\phi_1 = \frac{\cos\theta/2}{\sin\theta/2}$

$\Rightarrow \cot\phi_1 = \cot\theta/2 \quad \therefore \phi_1 = \theta/2$

$|||^{ly}$ (2) d. w.r.t θ

$\frac{dr}{d\theta} = -2a\sin\theta$

$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{-2a\sin\theta}{2a\cos\theta}$

$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = -\tan\theta$

$\Rightarrow \cot \phi_2 = -\tan\theta$

$\Rightarrow \cot \phi_2 = \cot(\pi/2 + \theta)$

$\phi_2 = \frac{\pi}{2} + \theta$

$\therefore |\phi_1 - \phi_2| = \left|\frac{\theta}{2} - \frac{\pi}{2} - \theta\right|$

$|\phi_1 - \phi_2| = \left|\frac{\pi}{2} + \frac{\theta}{2}\right|$

From ① and ②

$a(1-\cos\theta) = 2a\cos\theta$

$\Rightarrow (1-\cos\theta) = 2\cos\theta$

$\Rightarrow (1-\cos\theta) = 2\cos\theta$

$\Rightarrow 3\cos\theta = 1$

$\Rightarrow \cos\theta = 1/3$

$\Rightarrow \theta = \cos^{-1}(1/3)$

$\therefore |\phi_1 - \phi_2| = |\pi/2 + 1/2 \cos^{-1}(1/3)|$ //

$\therefore$ The Given two Curves or pair of curves are intersecting at $|\pi/2 + 1/2 \cos^{-1}(1/3)|$

⑤ $r = \frac{a}{1+\cos\theta}$ and $r = \frac{b}{1-\cos\theta}$

Given,

$r = \frac{a}{1+\cos\theta}$ or $r(1+\cos\theta) = a \longrightarrow$ ①

① d w.r.t θ

$$(1+\cos\theta)\frac{dr}{d\theta} - r\sin\theta = 0$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1+\cos\theta}$$

$$\Rightarrow \cot\phi_1 = \frac{2\sin\theta/2\cos\theta/2}{2\cos^2\theta/2}$$

$$\Rightarrow \cot\phi_1 = \tan\theta/2$$

$$\Rightarrow \cot\phi_1 = \cot(\pi/2 - \theta/2)$$

$$\therefore \phi_1 = (\pi/2 - \theta/2)$$

Similarly differentiate ② w.r.t. θ

$$r = \frac{b}{1-\cos\theta} \quad \text{or} \quad r(1-\cos\theta) = b \longrightarrow ②$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1-\cos\theta} = -\cot\theta/2$$

$$\Rightarrow \cot\phi_2 = -\frac{\theta}{2}$$

$$\therefore |\phi_2 - \phi_1| = \left|-\frac{\theta}{2} - \frac{\pi}{2} + \frac{\theta}{2}\right| = \frac{\pi}{2}$$

∴ The Given pair of curves are intersecting at the angle of $\frac{\pi}{2}$ or 90°

IV. Find the pedal equation For the Following polar curves.

① $r = ae^{\theta\cot\alpha}$

Let,

$r = ae^{\theta\cot\alpha} \longrightarrow ①$

① equation differentiate w.r.t. θ

$① \Rightarrow \frac{dr}{d\theta} = a \cdot e^{\theta\cot\alpha} \cdot \cot\alpha$

$\Rightarrow \frac{dr}{d\theta} = (a\cot\alpha)e^{\theta\cot\alpha}$

$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{(a\cot\alpha)e^{\theta\cot\alpha}}{r}$

$\Rightarrow \cot\phi = \frac{(a\cot\alpha)e^{\theta\cot\alpha}}{ae^{\theta\cot\alpha}}$

$\Rightarrow \cot\phi = \cot\alpha$

$\Rightarrow \phi = \alpha$

W.K.T the pedal equation

$p = r\sin\phi$

$\boxed{\Rightarrow p = r\sin\alpha}$

② $r^n = a^n\cos n\theta$

Given,

$r^n = a^n\cos n\theta \longrightarrow ①$

① equation differentiate w.r.t. θ

$① \Rightarrow nr^{n-1}\frac{dr}{d\theta} = a^n(-\sin n\theta)n$

$$\Rightarrow \frac{r^n}{r}\frac{dr}{d\theta} = -a^n \sin n\theta$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{-a^n \sin n\theta}{r^n}$$

$$\Rightarrow \cot\phi = \frac{-a^n \sin n\theta}{a^n \cos n\theta}$$

$$\Rightarrow \cot\phi = -\tan n\theta$$

W.K.T The pedal equation

$$\frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left[1+\tan^2 n\theta\right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\sec^2 n\theta$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2\cos^2 n\theta}$$

$$\Rightarrow p^2 = r^2\cos^2 n\theta$$

$$\Rightarrow p = r\cos n\theta$$

$$\Rightarrow p = r\left(\frac{r^n}{a^n}\right) \quad (\because ①)$$

$$\Rightarrow p = \frac{r^{n+1}}{a^n}$$

$$\boxed{\Rightarrow a^n p = r^{n+1}}$$

③ $\frac{2a}{r} = 1 + \cos\theta$

Given.

$$\frac{2a}{r} = 1+\cos\theta \quad \text{or} \quad 2a = r(1+\cos\theta) \longrightarrow (1)$$

(2) equation differentiate w.r.t. θ

$$(1) \Rightarrow (1+\cos\theta)\frac{dr}{d\theta} + r(-\sin\theta) = 0$$

$$\Rightarrow (1+\cos\theta)\frac{dr}{d\theta} - r\sin\theta = 0$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1+\cos\theta}$$

$$\Rightarrow \cot\phi = \frac{2\sin\theta/2\,\cos\theta/2}{2\cos^2\theta/2}$$

$$\Rightarrow \cot\phi = \tan\theta/2$$

$\therefore$ w.k.T The pedal equation

$$\frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}(1+\tan^2\theta/2)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}(\sec^2\theta/2)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2\cos^2\theta/2}$$

$$\Rightarrow p^2 = r^2\cos^2\theta/2$$

$$\Rightarrow p^2 = r^2\left(\frac{1+\cos\theta}{2}\right)$$

$$\Rightarrow 2p^2 = r^2(1+\cos\theta)$$

$$\Rightarrow 2p^2 = r^2\left(\frac{2a}{r}\right)$$

$$\boxed{\Rightarrow p^2 = ar}$$

(4) $r^m = a^m(\cos m\theta + \sin m\theta)$

Given

$$r^m = a^m[\cos m\theta + \sin m\theta] \longrightarrow (1)$$

(1) differentiate w.r.t. θ

$$(1) \Rightarrow m r^{m-1}\frac{dr}{d\theta} = a^m[-\sin m\theta(m) + \cos m\theta(m)]$$

$$\Rightarrow \frac{r^m}{r}\frac{dr}{d\theta} = \frac{a^m(\cos m\theta - \sin m\theta)}{r^m}$$

$$\Rightarrow \cot\phi = \frac{a^m(\cos m\theta - \sin m\theta)}{a^m(\cos m\theta + \sin m\theta)}$$

$$\Rightarrow \cot\phi = \frac{\cos m\theta - \sin m\theta}{\cos m\theta + \sin m\theta}$$

$$\therefore 1 + \cot^2\phi = \frac{1 + (\cos m\theta - \sin m\theta)^2}{(\cos m\theta + \sin m\theta)^2}$$

$$\Rightarrow 1 + \cot^2\phi = \frac{(\cos m\theta + \sin m\theta)^2 + (\cos m\theta - \sin m\theta)^2}{(\cos m\theta + \sin m\theta)^2}$$

$$\Rightarrow 1 + \cot^2\phi = \frac{2(\cos^2 m\theta + \sin^2 m\theta)}{\left(\frac{r^m}{a^m}\right)^2}$$

$$\Rightarrow 1 + \cot^2\phi = \frac{2}{r^{2m}/a^{2m}}$$

$$\Rightarrow 1 + \cot^2\phi = \frac{2a^{2m}}{r^{2m}}$$

$\therefore$ the pedal equation

$$\frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\cdot\frac{2a^{2m}}{r^{2m}}$$

$$\Rightarrow \frac{1}{p^2} = \frac{2a^{2m}}{r^{2m+2}}$$

$$\boxed{\Rightarrow 2a^{2m}p^2 = r^{2m+2}}$$

(5) $\frac{l}{r} = 1 + e\cos\theta$

Given,

$$\frac{l}{r} = 1 + e\cos\theta \longrightarrow (1)$$

differentiate w r t θ

$$(1) \Rightarrow (1+e\cos\theta)\frac{dr}{d\theta} + r(0 - e\sin\theta) = 0$$

$$\Rightarrow 1 + e\cos\theta\frac{dr}{d\theta} = re\sin\theta$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{e\sin\theta}{1+e\cos\theta} = \cot\phi$$

$$\therefore 1 + \cot^2\phi = 1 + \frac{e^2\sin^2\theta}{(1+e\cos\theta)^2}$$

$$= \frac{(1+e\cos\theta)^2 + e^2\sin^2\theta}{(1+e\cos\theta)^2}$$

$$= \frac{1 + 2e\cos\theta + e^2\cos^2\theta + e^2\sin^2\theta}{(1+e\cos\theta)^2}$$

$$= \frac{1 + 2e\cos\theta + e^2(1)}{(1+e\cos\theta)^2}$$

$$= \frac{1 + 2e\cos\theta + e^2}{(1+e\cos\theta)^2}$$

$$\Rightarrow 1 + \cot^2\phi = \frac{1 + e^2 + 2\left(\frac{l}{r} - 1\right)}{(l/r)^2}$$

$$= \frac{1 + e^2 + \frac{2l}{r} - 2}{l^2/r^2}$$

$$= \frac{e^2 + \frac{2l}{r} - 1}{l^2/r^2}$$

$$1 + \cot^2\phi = \frac{r^2(e^2 + 2l/r - 1)}{l^2}$$

$\therefore$ the pedal equation

$$\frac{1}{p^2} = \frac{1}{r^2}(1 + \cot^2\phi)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} \cdot r^2 \left(\frac{e^2 + 2l/r - 1}{l^2}\right)$$

$$\boxed{\Rightarrow \frac{1}{p^2} = \frac{e^2 + 2l/r - 1}{l^2}}$$ //

Radius of Curvature

Generally the Curvature of any curve Can be denoted by 'k' and the reciprocal of the curvature will be called as the radius of curvature and it will be denoted as,

$$\rho = \frac{1}{K}$$

Note:-

1. the radius of Curvature can be evaluated in the Cartisian form by the following formula.

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}, \quad y_2 \neq 0$$

where $y_1 = \frac{dy}{dx}$, $y_2 = \frac{d^2y}{dx^2}$ at any point P

In other way $\rho = \frac{(1+x_1^2)^{3/2}}{x_2}, \quad x_2 \neq 0$

where $x_1 = \frac{dx}{dy}$, $x_2 = \frac{d^2x}{dy^2}$ at any point P

2. The radius of curvature for the polar curve can be evaluated as,

$\rho = r\frac{dr}{dp}$, where $f(p, r, c)$ is the pedal equation.

V

1. Find the radius of curvature of the curve $y = a\log(\sec(x/a))$ at any point

Given,

$y = a\log(\sec(x/a))$

differentiate ① w.r.t. x

$$① \Rightarrow \frac{dy}{dx} = a \cdot \frac{1}{\sec(x/a)} \sec(x/a)\tan(x/a) \cdot 1/a$$

$\Rightarrow \frac{dy}{dx} = \tan(x/a) = y_1 \longrightarrow ②$

differentiate ② w.r.t. x

② $\Rightarrow \frac{d^2y}{dx^2} = \sec^2(x/a) \cdot 1/a$

$\Rightarrow y_2 = \frac{d^2y}{dx^2} = \frac{1}{a}\sec^2(x/a)$

$\therefore$ w.K.T

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$\Rightarrow \rho = \frac{(1+\tan^2(x/a))^{3/2}}{1/a \sec^2(x/a)}$$

$$\Rightarrow \rho = \frac{a(\sec^2(x/a))^{3/2}}{\sec^2(x/a)}$$

$$\Rightarrow \rho = \frac{a \sec^3(x/a)}{\sec^2(x/a)}$$

$$\boxed{\Rightarrow \rho = a\sec(x/a)}$$

2. Find the radius of the curvature of the curve $x^3+y^3 = 3axy$ at the point $P(3a/2, 3a/2)$.

Given,

$x^3 + y^3 = 3axy \longrightarrow ①$

differentiate w r t x

① $\Rightarrow 3x^2 + 3y^2\frac{dy}{dx} = 3a\left[1.y + x\frac{dy}{dx}\right]$

$\Rightarrow x^2 + y^2 y_1 = a[y + xy_1]$

$\Rightarrow x^2 + y^2 y_1 - ay - axy_1 = 0$

$\Rightarrow (y^2 - ax) y_1 + (x^2 - ay) = 0$

$\Rightarrow (y^2 - ax) y_1 + (x^2 - ay) = 0$

$\Rightarrow (y^2 - ax) y_1 = ay - x^2$

$$\Rightarrow y_1 = \frac{ay - x^2}{y^2 - ax} \longrightarrow (2)$$

$$\therefore (y_1)_p = \frac{a\left(\frac{3a}{2}\right) - \left(\frac{3a}{2}\right)^2}{\left(\frac{3a}{2}\right)^2 - a\left(\frac{3a}{2}\right)} = -\frac{\left(\frac{3a}{2}\right)^2 - a\left(\frac{3a}{2}\right)}{\left(\frac{3a}{2}\right)^2 - a\left(\frac{3a}{2}\right)} = -1$$

differentiate (2) w.r.t x

$$(2) \Rightarrow y_2 = \frac{d^2y}{dx^2} = \frac{(y^2 - ax)(ay_1 - 2x) - (ay - x^2)(2yy_1 - a)}{(y^2 - ax)^2}$$

$$(y_2)_p = \frac{\left[\left(\frac{3a}{2}\right)^2 - 4\left(\frac{3a}{2}\right)\right] a(-1) - 2\left(\frac{3a}{2}\right) - 4\left(\frac{3a}{2}\right) - \left(\frac{3a}{2}\right)^2 \left[2\left(\frac{3a}{2}\right)(-1) - a\right]}{\left(\frac{3a}{2}\right)^2 - a\left(\frac{3a}{2}\right)^2}$$

$$\Rightarrow y_2 = \frac{\left(\frac{3a^2}{4}\right)(-4a) - \left(-\frac{3a^2}{4}\right)(-4a)}{\left(\frac{3a^2}{4}\right)^2}$$

$$\Rightarrow y_2 = \frac{-3a^3 - 3a^3}{\frac{9a^4}{16}}$$

$\Rightarrow y_2 = \dfrac{-6a^3 \times 16}{9a^4} = \dfrac{-32}{3a}$

$\therefore \rho = \left| \dfrac{(1+y_1^2)^{3/2}}{y_2} \right| = \left| \dfrac{(1+(-1)^2)^{3/2}}{-32/3a} \right|$

$\Rightarrow \rho = \dfrac{2^{3/2}}{32/3a}$

$\Rightarrow \rho = \dfrac{2 \cdot 2^{1/2} \times 3a}{32}$

$\Rightarrow \rho = \dfrac{3\sqrt{2}a}{16}$

$\Rightarrow \rho = \dfrac{3\sqrt{2} \cdot a}{8 \times \sqrt{2} \times \sqrt{2}}$

$\boxed{\Rightarrow \rho = \dfrac{3a}{8\sqrt{2}}}$

③ Find the radius of curvature of the curve $a^2y = x^2 - a^2$ at the point where the curve meets at x-axis

Given curve $a^2y = x^2 - a^2$ and the curve meet the x axis at the point 'P' and its y coordinate becomes zero.

$\therefore$ the required point $P(a, 0)$

differentiate equation w r t x

$a^2y = x^2 - a^2 \longrightarrow ①$

$① \Rightarrow a^2 y_1 = 2x - 0$

$\Rightarrow y_1 = \dfrac{2x}{a^2} \longrightarrow ②$

$\therefore (y_1)_p = \dfrac{2(a)}{a^2} = \dfrac{2}{a} \longrightarrow ③$

differentiate ③ w.r.t x

$③ \Rightarrow y_2 = \dfrac{d^2y}{dx^2} = \dfrac{2}{a^2}$

$(y_2)_p = \dfrac{2}{a^2} \neq 0$

$$\rho = \left| \frac{(1-y_1^2)^{3/2}}{y_2} \right|$$

$$\rho = \frac{\left(1+(2/a)^2\right)^{3/2}}{2/a^2}$$

$$\Rightarrow \rho = \frac{(1+4/a^2)^{3/2}}{2/a^3}$$

$$\Rightarrow \rho = \frac{a^2\left(\frac{a^2+4}{a^2}\right)^{3/2}}{2}$$

$$\Rightarrow \rho = \frac{\frac{a^2(a^2+4)^{3/2}}{a^3}}{2}$$

$$\Rightarrow \boxed{\rho = \frac{(a^2+4)^{3/2}}{2a}}$$

④. Find the radius of the curvature of the curve $a^2y = x^3 - a^3$ and meets at x axis.

Given,

$a^2y = x^3 - a^3 \longrightarrow ①$

differentiate w r t. x

$① \Rightarrow a^2y_1 = 3x^2 \longrightarrow ②$

or

$y_1 = \frac{3x^2}{a^2} \longrightarrow ②$

$\therefore (y_1)_p = \frac{3(a^2)}{a^2} = 3$

differentiate ② w. r t x

$② \Rightarrow y_2 = \frac{d^2y}{dx^2} = \frac{6x}{a^2}$

$(y_2)_p = \frac{6(a^2)}{a^3} = \frac{6}{a}$

$\Rightarrow \rho = \left| \frac{(1+y_1^2)^{3/2}}{y_2} \right|$

$\Rightarrow \rho = \frac{(1+3^2)^{3/2}}{6/a}$

$\Rightarrow \rho = \frac{(1+9)^{3/2}}{6/a}$

$\Rightarrow \rho = \frac{a \cdot 10^{3/2}}{6}$

$$\Rightarrow \rho = \frac{a \cdot 10 \cdot 10^{1/2}}{6}$$

$$\Rightarrow \boxed{\rho = \frac{5\sqrt{10} \cdot a}{3}}$$

⑤ Find the radius of the curvature of the curve $x^2y = a(x^2+y^2)$ at Point $P(-2a, 2a)$

Given,

$$x^2y = a(x^2+y^2) \longrightarrow ①$$

differentiate ① w.r.t y

$$① \Rightarrow x^2(1) + y\,2x\frac{dx}{dy} = a\left(2x\frac{dx}{dy} + 2y\right)$$

$$\Rightarrow x^2 + 2xy \cdot x_1 = 2a x_1 + 2ay$$

$$\Rightarrow 2xy \cdot x_1 - 2ax \cdot x_1 = 2ay - x^2$$

$$\Rightarrow 2x x_1(y-a) = 2ay - x^2$$

$$\Rightarrow x_1 = \frac{2ay - x^2}{2x(y-a)} = \frac{2ay - x^2}{2xy - 2ax}$$

$$\therefore (x_1)_P = \frac{2a(2a) - (-2a)^2}{2(-2a)[2a-a]}$$

$$\Rightarrow (x_1)_P = 0$$

differentiate ② w.r.t y

$$② \Rightarrow x_2 = \frac{d^2x}{dy^2} = \frac{(2xy-2ax)(2a - 2x \cdot x_1) - (2a - 2x \cdot x_1) - (2ay - x^2)[2x + xx_1y - 2ax_1]}{[2x(y-a)]^2}$$

$$\therefore (x_2)_p = \frac{[-8a^2+4a^2][2a-0]-0}{[-4a(a)]^2}$$

$$\Rightarrow x_2 = \frac{-8a^3}{16a^4}$$

$$\Rightarrow x_2 = \frac{-1}{2a} \neq 0$$

$$\therefore \rho = \left| \frac{(1+x_1^2)^{3/2}}{x_2} \right| = \left| \frac{(1+0)^{3/2}}{-1/2a} \right| = 2a$$ //

⑥ Find the radius of the curvature of the polar curve $r = a(1+\cos\theta)$

Given,

$r = a(1+\cos\theta) \longrightarrow ①$

differentiate ① w r t θ

① $\Rightarrow \frac{dr}{d\theta} = a(0-\sin\theta) = -a\sin\theta$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{-a\sin\theta}{a(1+\cos\theta)}$$

$$\Rightarrow \cot\phi = \frac{-2\sin\theta/2 \cos\theta/2}{2\cos^2\theta/2}$$

$$\Rightarrow \cot\phi = -\tan(\theta/2)$$

W.K.T the pedal equation

$$\frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left(1 + \tan^2 \theta/2\right)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} \sec^2 \theta/2$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2 \cos^2 \theta/2}$$

$$\Rightarrow p^2 = r^2 \cos^2 \theta/2$$

$$\Rightarrow p^2 = r^2\left(\frac{1+\cos\theta}{2}\right)$$

$$\Rightarrow p^2 = \frac{r^2}{2}\left(\frac{r}{a}\right)$$

$$\Rightarrow 2a p^2 = r^3 \longrightarrow (2)$$

differentiate (2) w r t p

$$(2) \Rightarrow 2a \cdot 2p = 3r^2 \frac{dr}{dp}$$

$$\Rightarrow 4ap = 3r^2 \frac{dr}{dp}$$

$$\Rightarrow \frac{dr}{dp} = \frac{4ap}{3r^2}$$

$$\Rightarrow r\frac{dr}{dp} = \frac{4ap}{3r^2}$$

$$\Rightarrow r\frac{dr}{dp} = \frac{4ap}{3r}$$

$$\Rightarrow \rho = \frac{4ap}{3r}$$

$$\Rightarrow \rho^2 = \frac{16a^2 p^2}{9r^2}$$

$$\Rightarrow \rho^2 = \frac{16a^2}{9} \cdot \frac{1}{r^2}\left(\frac{r^3}{2a}\right)$$

$$\Rightarrow \rho^2 = \left(\frac{8a}{9}\right) r$$

$$\boxed{\Rightarrow \rho^2 \propto r}$$

(7) Show that for the curve $r(1-\cos\theta) = 2a$ or $\frac{2a}{r} = (1-\cos\theta)$, and ρ^2 varies as r^3 $[\rho^2 \propto r^3]$

Given,

$r(1-\cos\theta) = 2a \longrightarrow (1)$

differentiate (1) w.r.t θ

$$(1) \Rightarrow (1-\cos\theta)\frac{dr}{d\theta} + r(0+\sin\theta) = 0$$

$$\Rightarrow (1-\cos\theta)\frac{dr}{d\theta} = -r\sin\theta$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1-\cos\theta}$$

$$\Rightarrow \cot\phi = \frac{-2\sin\theta/2 \cos\theta/2}{2\sin^2\theta/2}$$

$$\Rightarrow \cot\phi = -\cot(\theta/2)$$

$\therefore$ W.K.T the pedal equ.

$$\frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\theta/2)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}(\operatorname{cosec}^2\theta/2)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2\sin^2\theta/2}$$

$$\Rightarrow p^2 = r^2\sin^2\theta/2$$

$$\Rightarrow p^2 = r^2\left(\frac{1-\cos\theta}{2}\right)$$

$$\Rightarrow p^2 = \frac{r^2}{2}\left(\frac{2a}{r}\right)$$

$$\Rightarrow p^2 = ar \longrightarrow (2)$$

differentiate (2) w.r.t. p

$$(2) \Rightarrow 2p = a\frac{dr}{dp}$$

$$\Rightarrow \frac{dr}{dp} = \frac{2p}{a}$$

$$\Rightarrow r\frac{dr}{dp} = \frac{2rp}{a}$$

$$\Rightarrow \rho = \frac{2rp}{a}$$

$$\Rightarrow \rho^2 = \frac{4r^2p^2}{a^2}$$

$$\Rightarrow \rho^2 = \frac{4r^2}{a^2}(ar)$$

$$\Rightarrow \rho^2 = \left(\frac{4}{a}\right) r^3$$

$$\boxed{\Rightarrow \rho^2 \propto r^3}$$

⑧ For the cardiant, $r = a(1-\cos\theta)$, show that $\frac{\rho^2}{r}$ is constant

Given,

$$r = a(1-\cos\theta) \rightarrow ①$$

differentiate ① w.r.t θ

$$\frac{dr}{d\theta} = \sin\theta \cdot a$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{a\, 2\sin\theta/2 \cos\theta/2}{a \cdot 2\sin^2\theta/2}$$

$$\frac{1}{r}\frac{dr}{d\theta} = \cot\theta/2$$

$$\cot\phi = \cot\theta/2$$

$$\phi = \theta/2$$

$\therefore$ W.K.T the pedal equation

$$p = r\sin(\theta/2)$$

$$p^2 = r^2\sin^2(\theta/2)$$

$$p^2 = r^2\left(\frac{1-\cos\theta}{2}\right)$$

$$\Rightarrow p^2 = \frac{r}{2}\left(\frac{r}{a}\right)$$

$$\Rightarrow p^2 = \frac{r^3}{2a}$$

then differentiate w.r.t p

$$2p = \frac{3r^2}{2a}\frac{dr}{dp}$$

$$r\frac{dr}{dp} = \frac{2p}{\frac{3r}{2a}}$$

$$\Rightarrow \rho = \frac{2p \times 2a}{3r}$$

$$\Rightarrow \rho^2 = \frac{16a}{9r^2}\left(\frac{r^3}{2a}\right)$$

$$\Rightarrow \boxed{\frac{\rho^2}{r} = \frac{8a}{9}}$$ //

⑨ Find the radius of curvature for the curve $\theta = \frac{\sqrt{r^2-a^2}}{a} - \cos^{-1}\left(\frac{a}{r}\right)$ at any point on it

Given,

$$\theta = \frac{\sqrt{r^2-a^2}}{a} - \cos^{-1}\left(\frac{a}{r}\right) \longrightarrow ①$$

differentiate ① w.r.t. r

$$① \Rightarrow \frac{d\theta}{dr} = \frac{1}{a}\frac{1}{2\sqrt{r^2-a^2}}(2r) - \left(\frac{-1}{\sqrt{1-(a/r)^2}}\right)\frac{d}{dr}\left(\frac{a}{r}\right)$$

$$\Rightarrow \frac{d\theta}{dr} = \frac{r}{a\sqrt{r^2-a^2}} + \frac{1}{\sqrt{1-a^2/r^2}}\, a\left(\frac{-1}{r^2}\right)$$

$$= \frac{r}{a\sqrt{r^2-a^2}} - \frac{ar}{\sqrt{r^2-a^2}}\left(\frac{1}{r^2}\right)$$

$$= \frac{r}{a\sqrt{r^2-a^2}} - \frac{a}{\sqrt{r^2-a^2}}$$

$$\Rightarrow \frac{d\theta}{dr} = \frac{1}{\sqrt{r^2-a^2}}\left[\frac{r}{a} - \frac{a}{r}\right]$$

$$= \frac{1}{\sqrt{r^2-a^2}}\left[\frac{r^2-a^2}{ar}\right]$$

$$= \frac{1}{\sqrt{r^2-a^2}}\,\frac{\sqrt{r^2-a^2}\,\sqrt{r^2-a^2}}{ar}$$

$$\frac{d\theta}{dr} = \frac{\sqrt{r^2-a^2}}{ar}$$

$$\Rightarrow \frac{dr}{d\theta} = \frac{ar}{\sqrt{r^2-a^2}}$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{a}{\sqrt{r^2-a^2}}$$

$$\Rightarrow \cot\phi = \frac{a}{\sqrt{r^2-a^2}}$$

$$\Rightarrow \cot\phi = \frac{a^2}{r^2-a^2}$$

$$\Rightarrow 1+\cot^2\phi = \frac{1+a^2}{r^2-a^2} = \frac{r^2-a^2+a^2}{r^2-a^2} = \frac{r^2}{r^2-a^2}$$

∴ the pedal equation

$$\frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi) = \frac{1}{r^2}\frac{r^2}{(r^2-a^2)}$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2-a^2}$$

$$\Rightarrow p^2 = r^2 - a^2 \longrightarrow ②$$

differentiate ② w.r.t. p

$$② \Rightarrow 2p = 2r\frac{dr}{dp}$$

$$\Rightarrow r\frac{dr}{dp} = p$$

$$\boxed{\Rightarrow \rho = \sqrt{r^2-a^2}}$$

⑩ Find the radius of curvature $r^n = a^n(\sin n\theta)$

Given,

$$r^n = a^n(\sin n\theta) \longrightarrow ①$$

differentiate w r t r

$$n r^{n-1} \frac{dr}{d\theta} = a^n (\cos n\theta)$$

$$\frac{r^n}{r} \frac{dr}{d\theta} = \frac{a^n \cos n\theta}{r}$$

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{a^n \cos n\theta}{a^n \sin n\theta}$$

$$\cot \phi = \cot n\theta$$

∴ The pedal equation

$$\frac{1}{p^2} = \frac{1}{r^2} (1 + \cot^2 \phi)$$

$$\frac{1}{p^2} = \frac{\operatorname{cosec}^2 n\theta}{r^2}$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2 \sin^2 n\theta}$$

$$\Rightarrow p^2 = r^2 \sin^2 n\theta$$

$$\Rightarrow p^2 = r \sin n\theta$$

$$\Rightarrow p^2 = r \frac{r^n}{a^n} = \frac{r^{n+1}}{a^n}$$

$$\Rightarrow a^n p = r^{n+1} \longrightarrow (2)$$

differentiate (2) w r t p

$$(2) \Rightarrow a^n . 1 = (n+1) r^n \frac{dr}{dp}$$

$$\Rightarrow r^n \frac{dr}{dp} = \frac{a^n}{n+1}$$

$$\Rightarrow r . r^{n-1} \frac{dr}{dp} = \frac{a^n}{n+1}$$

$$\Rightarrow r\frac{dr}{dp} = \left(\frac{a^n}{n+1}\right)\cdot\left(\frac{1}{r^{n-1}}\right)$$

$$\Rightarrow \rho = \left(\frac{a^n}{n+1}\right)\left(\frac{1}{r^{n-1}}\right)$$

$$\boxed{\Rightarrow \rho \propto \frac{1}{r^{n-1}}}$$

(11) Find the radius of Curvature $r^n = a^n(\cos n\theta)$

Given,

$$r^n = a^n(\cos n\theta) \longrightarrow (1)$$

differentiate (1) w.r.t r

$$(1) \Rightarrow \quad n\, r^{n-1}\frac{dr}{d\theta} = -a^n \cdot n \sin^n\theta$$

$$\frac{1}{r}\frac{r^n}{r}\frac{dr}{d\theta} = \frac{-a^n \sin n\theta}{a^n \cos n\theta}$$

$$\frac{1}{r}\frac{dr}{d\theta} = -\frac{a^n \sin n\theta}{a^n \cos n\theta}$$

$$\cot\phi = -\tan n\theta$$

$$\cot\phi = \cot(\pi/2 + n\theta)$$

$$\phi = \frac{\pi}{2} + n\theta$$

$\therefore$ W K T the pedal equation

$\Rightarrow \quad p = r \sin\phi$

$\Rightarrow \quad p = r \sin\left(\pi/2 + n\theta\right)$

$\Rightarrow \quad p = r \cos n\theta$

$\Rightarrow p = r\left(\frac{r^n}{a^n}\right)$

$\Rightarrow \quad p = \frac{r^{n+1}}{a^n}$

$\boxed{\Rightarrow \quad a^n p = r^{n+1}} \longrightarrow (2)$

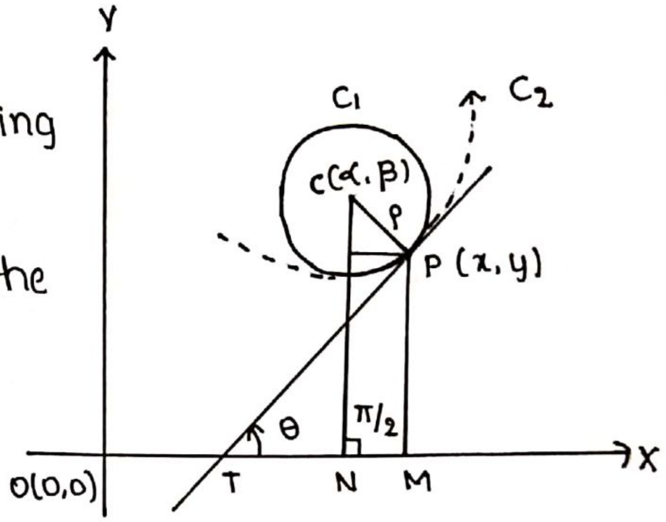
Then differentiate w r t p

$a^n = (n+1) r^n \frac{dr}{dp}$

$\frac{a^n}{(n+1)}\left(\frac{1}{r^{n-1}}\right) = \rho$

$\boxed{\Rightarrow \rho \propto \frac{1}{r^{n-1}}}$ //

The Centre of Curvature , Invaluates and Evaluates



let C_1 and C_2 be the two Smooth curves passing through the Same point $P(x,y)$ the Centre of the Curve C_1 is $C(\alpha,\beta)$ Called as the Centre of Curvature, and it's radius ρ is Called circle curvature the curve C_2 is Called as the evaluate of the invaluate C_1. the evaluate of an invaluate can be derived by the locus of the centre of the Curvature $C(\alpha, \beta)$.

The Centre of Curvature $C(\alpha,\beta)$ can be evaluated by using the following formulas

$$\alpha = x - \rho \sin\psi$$

$$\beta = y + \rho \cos\psi$$

$$\Rightarrow \alpha = \frac{x - y_1(1+y_1^2)}{y_2} , \quad \beta = \frac{y + (1+y_1^2)}{y_2}$$

where , $y_1 = \frac{dy}{dx}$, $y_2 = \frac{dy^2}{dx^2}$

VIP ①. Show that the evaluate of parabola, $y^2 = 4ax$ is $27ay^2 = 4(x-2a)^3$

Given.

$y^2 = 4ax \rightarrow ①$

$\Rightarrow y = 2a^{1/2} x^{1/2}$

differentiate ① w.r.t. x

$① \Rightarrow 2y \frac{dy}{dx} = 4a$

$\Rightarrow y \frac{dy}{dx} = 2a$

$\Rightarrow \frac{dy}{dx} = y_1 = \frac{2a}{y} \longrightarrow ②$

differentiate ② w.r.t. x

$② \Rightarrow \frac{d^2y}{dx^2} = y_2 = \frac{y(a) \; 2ay_1}{y^2}$

$\Rightarrow y_2 = \frac{-2ay_1}{y^2}$

$\Rightarrow y_2 = \frac{-2a(2a/y)}{y^2}$

$\Rightarrow y_2 = \frac{-4a^2}{y^3}$

$\Rightarrow y_2 = \frac{-4a^2}{(2a^{1/2} \cdot x^{1/2})^3}$

$\Rightarrow y_2 = \frac{-4a^3}{8a^{3/2} x^{3/2}}$

$\Rightarrow y_2 = \frac{-a^{1/2}}{2x^{3/2}}$

$\therefore \quad y_1 = \dfrac{2a}{2a^{1/3}x^{1/2}}$

$\Rightarrow \quad y_1 = \dfrac{a^{1 \times 1/2}}{x^{1/2}}$

$\therefore$ W.K.T

$$\alpha = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$\Rightarrow \alpha = x - \frac{\left(\frac{a^{1/2}}{x^{1/2}}\right)(1+a/x)}{\frac{-a^{1/2}}{2x^{3/2}}}$$

$$\Rightarrow \alpha = \frac{x - 2a^{1/2}x^{3/2}\left(\frac{x+a}{x}\right)}{-a^{1/2}x^{1/2}}$$

$$\Rightarrow \alpha = \frac{x + 2x \cdot x^{1/2}\left(\frac{x+a}{x}\right)}{x^{1/2}}$$

$$\Rightarrow \alpha = x + 2(x+a)$$

$$\Rightarrow \alpha = 3x + 2a$$

$$\Rightarrow 3x = \alpha - 2a$$

$$\Rightarrow x = \frac{(\alpha - 2a)}{3}$$

$$\Rightarrow x^3 = \frac{(\alpha-2a)^3}{27} \longrightarrow (3)$$

$$\Rightarrow \beta = \frac{y + (1+y_1)^2}{y_2} = \frac{2a^{1/2}x^{1/2} + \left(1+\frac{a}{x}\right)}{\frac{-a^{1/2}}{2x^{3/2}}}$$

$$\Rightarrow \beta = 2a^{1/2}x^{1/2} - \frac{2x^{3/2}\left(x+\frac{a}{x}\right)}{a^{1/2}}$$

$$\Rightarrow \beta = 2a^{1/2}x^{1/2} - \frac{2x x^{1/2}\left(\frac{x+a}{x}\right)}{a^{1/2}}$$

$$\Rightarrow \beta = 2a^{1/2}x^{1/2} - \frac{2x^{1/2}(x+a)}{a^{1/2}}$$

$$\Rightarrow \beta = \frac{1}{a^{1/2}}\left| 2ax^{1/2} - 2x^{3/2} - 2ax^{1/2} \right|$$

$$\Rightarrow \beta = \frac{-2x^{3/2}}{a^{1/2}}$$

$$\Rightarrow -2x^{3/2} = \beta a^{1/2}$$

$$\Rightarrow x^{3/2} = \frac{-\beta^{a/2}}{2}$$

$$\Rightarrow (x^{3/2})^2 = \left[\frac{-\beta a^{1/2}}{2}\right]^2$$

$$\Rightarrow x^3 = \frac{\beta^2 a}{4} \longrightarrow (4)$$

From (3) and (4)

$$\frac{(\alpha-2a)^3}{27} = \frac{\beta^2 a}{4}$$

$$\Rightarrow 4(\alpha-2a)^3 = 27a\beta^2 \longrightarrow (5)$$

$\therefore$ the locus of equation (5) is

$$\boxed{4(x-2a)^3 = 27ay^2}$$

② Show that evaluate of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $(ax)^{2/3}+(by)^{2/3}=(a^2-b^2)^{2/3}$

Given.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \longrightarrow ①$$

Let $x = a\cos\theta$, $y = b\sin\theta$

$$\therefore \frac{dx}{d\theta} = -a\sin\theta , \quad \frac{dy}{d\theta} = b\cos\theta$$

$$y_1 = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cot\theta \longrightarrow ②$$

differentiate ② w r t x

$$\Rightarrow y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$\Rightarrow y_2 = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx}$$

$$\Rightarrow y_2 = \frac{d}{d\theta}\left(-\frac{b}{a}\cot\theta\right)\cdot\frac{1}{dx/d\theta}$$

$$\Rightarrow y_2 = \frac{-b}{a}(-\csc^2\theta)\cdot\frac{1}{-a\sin\theta}$$

$$\Rightarrow y_2 = -\frac{b}{a^2}\frac{1}{\sin^2\theta}\cdot\frac{1}{\sin\theta}$$

$$\Rightarrow y_2 = \frac{-b}{a^2\sin^2\theta}$$

$\therefore$ W.K.T

$$\alpha = \frac{x - y_1(1 + y_1^2)}{y_2}$$

$$\alpha = \frac{a\cos\theta - (-b/a \cot\theta)\left(1 + \frac{b^2}{a^2}\cot^2\theta\right)}{-b/a^2 \sin^3\theta}$$

$$\Rightarrow \alpha = \frac{a\cos\theta - \frac{b}{a}\frac{\cos\theta}{\sin\theta}\left[1 + \frac{b^2}{a^2}\frac{\cos^2\theta}{\sin^2\theta}\right]}{\frac{b}{a^2\sin^3\theta}}$$

$$\Rightarrow \alpha = a\cos\theta - (a\sin^3\theta)\left(\frac{\cos\theta}{\sin\theta}\right)\left(\frac{a^2\sin^2\theta + b^2\cos^2\theta}{a^2\sin^2\theta}\right)$$

$$\Rightarrow \alpha = a\cos\theta - \frac{\cos\theta}{a}\left[a^2(1-\cos^2\theta) + b^2\cos^2\theta)\right]$$

$$\Rightarrow \alpha = a\cos\theta - \frac{\cos\theta}{a}\left[a^2 - a^2\cos^2\theta + b^2\cos^2\theta\right]$$

$$\Rightarrow \alpha = a\cos\theta - a\cos\theta + \left[a\cos^3\theta - \frac{b^2}{a}\cos^3\theta\right]$$

$$\Rightarrow \alpha = \left(a - \frac{b^2}{a}\right)\cos^3\theta$$

$$\Rightarrow \alpha = \left(\frac{a^2 - b^2}{a}\right)\cos^3\theta$$

$$\Rightarrow \cos^3\theta = \frac{a\alpha}{a^2 - b^2}$$

$$\Rightarrow (\cos^3\theta)^{2/3} = (a\alpha)^{2/3} / (a^2 - b^2)^{2/3}$$

$$\Rightarrow \cos^2\theta = \frac{(a\alpha)^{2/3}}{(a^2-b^2)^{2/3}} \longrightarrow ③$$

$$\beta = \frac{y + (1+y_1^2)}{y_2}$$

$$\Rightarrow \beta = \frac{b\sin\theta + \left(1 + \frac{b^2}{a^2}\cot^2\theta\right)}{-b/a^2 \sin^3\theta}$$

$$\Rightarrow \beta = b\sin\theta - \frac{a^2\sin^3\theta}{b} - b\sin\theta + b\sin^3\theta$$

$$\Rightarrow \beta = \left(b - \frac{a^2}{b}\right)\sin^3\theta$$

$$\Rightarrow \beta = \left(\frac{b^2-a^2}{b}\right)\sin^3\theta$$

$$\Rightarrow \sin^3\theta = \frac{b\beta}{b^2-a^2}$$

$$\Rightarrow \sin^3\theta = \frac{b\beta}{-(a^2-b^2)}$$

$$\Rightarrow \sin^6\theta = \frac{(b\beta)^2}{(a^2-b^2)^{2/3}}$$

$$\Rightarrow \sin^2\theta = \frac{(b\beta)^{2/3}}{(a^2-b^2)^{2/3}} \longrightarrow ④$$

$$③ + ④ = \cos^2\theta + \sin^2\theta = \frac{(a\alpha)^{2/3}}{(a^2-b^2)^{2/3}} + \frac{(b\beta)^{2/3}}{(a^2-b^2)^{2/3}}$$

$$\Rightarrow (a\alpha)^{2/3} + (b\beta)^{2/3} = (a^2-b^2)^{2/3}$$

$\therefore$ the locus of (α, β)

$$(ax)^{2/3} + (by)^{2/3} = (a^2-b^2)^{2/3}$$ //

③ Show that the evaluate of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $(ax)^{2/3} - (by)^{2/3} = (a^2+b^2)^{2/3}$

Given,

$x = a\sec\theta$, $y = b\tan\theta$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \longrightarrow ①$$

$$\frac{dx}{d\theta} = a\sec\theta\tan\theta, \quad \frac{dy}{d\theta} = b\sec^2\theta$$

$$\therefore y_1 = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b\sec\theta}{a\tan\theta}$$

$$y_1 = \frac{b}{a}\operatorname{cosec}\theta$$

$$y_2 = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx}$$

$$= \frac{d}{d\theta}\left(\frac{b}{a}\operatorname{cosec}\theta\right)\frac{1}{dx/d\theta}$$

$$= \frac{b}{a}(-\operatorname{cosec}\theta \cdot \cot\theta) \cdot \frac{1}{a\sec\theta\tan\theta}$$

$$= \frac{-b}{a^2} \frac{1}{\sin\theta} \frac{\cos\theta}{\sin\theta} \cdot \cos\theta \cdot \frac{\cos\theta}{\sin\theta}$$

$$y_2 = \frac{-b}{a^2} \frac{\cos^3\theta}{\sin^3\theta}$$

$$\therefore \alpha = \frac{x - y_1(1+y_1^2)}{y_2}$$

$$\alpha = \frac{a\sec\theta - \left(\frac{b}{a}\operatorname{cosec}\theta\right)\left[1 + \frac{b^2}{a^2}\operatorname{cosec}^2\theta\right]}{-\frac{b}{a^2}\frac{\cos^3\theta}{\sin^3\theta}}$$

$$\alpha = a\sec\theta + \frac{b}{a}\frac{1}{\sin\theta}\frac{a^2\tan^3\theta}{b\cos^3\theta}\left[1 + \frac{b^2}{a^2}\cdot\frac{1}{\sin^2\theta}\right]$$

$$\alpha = a\sec\theta + \frac{a\sin^2\theta}{\cos^3\theta}\left[\frac{a^2\sin^2\theta + b^2}{a^2\sin^2\theta}\right]$$

$$\alpha = a\sec\theta + \frac{1}{a\cos^3\theta}\left[a^2\sin^2\theta + b^2\right]$$

$$\alpha = a\sec\theta + \frac{a\sin^2\theta}{\cos^3\theta} + \frac{b^2}{a\cos^3\theta}$$

$$\alpha = a\sec\theta + a\tan^2\theta\sec\theta + \frac{b^2}{a}\sec^3\theta$$

$$\alpha = a\sec\theta + a\sec\theta(\sec^2\theta - 1) + \frac{b^2}{a}\sec^3\theta$$

$$\alpha = a\sec\theta + a\sec^3\theta - a\sec\theta + \frac{b^2}{a}\sec^3\theta$$

$$\alpha = \left(a + \frac{b^2}{a}\right)\sec^3\theta$$

$$\alpha = \left(\frac{a^2+b^2}{a}\right)\sec^3\theta$$

$$\sec^3\theta = \frac{a\alpha}{a^2+b^2}$$

$$\sec^2\theta = \frac{(a\alpha)^{2/3}}{(a^2+b^2)^{2/3}} \longrightarrow (4)$$

Similarlly $\beta = y + \frac{(1+y_1^2)}{y_2}$

$$\tan^2\theta = \frac{(b\beta)^{3/2}}{(a^2+b^2)^{2/3}} \longrightarrow (5)$$

$$(4) + (5) = 1 = \frac{(a\alpha)^{2/3} - (b\beta)^{2/3}}{(a^2+b^2)^{2/3}}$$

$$(a^2+b^2)^{2/3} = (ax)^{2/3} - (by)^{2/3} \;//$$

A. Find the pedal equation.

① $r^n = a^n \sin n\theta$

Given,

$r^n = a^n \sin n\theta \longrightarrow ①$

① equation differentiate w.r.t. θ

$$① \Rightarrow n r^{n-1} \frac{dr}{d\theta} = a^n \cos n\theta . n$$

$$\Rightarrow \frac{r^n}{r} \frac{dr}{d\theta} = a^n \cos n\theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a^n \cos n\theta}{r^n}$$

$$\Rightarrow \cot\phi = \frac{a^n \cos n\theta}{a^n \sin n\theta}$$

$$\Rightarrow \cot\phi = \cot n\theta$$

$$\Rightarrow \phi = n\theta$$

We know that, the pedal equation

$$\frac{1}{p^2} = \frac{1}{r^2}(1 + \cot^2\phi)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}(1 + \cot^2 n\theta)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} \operatorname{cosec}^2 n\theta$$

$$\Rightarrow p^2 = r^2 \sin^2 n\theta$$

$$\Rightarrow p = r \sin n\theta$$

$$\Rightarrow p = r\left(\frac{r^n}{a^n}\right)$$

$$\boxed{\Rightarrow p = \frac{r^{n+1}}{a^n}}$$

B. Find the pedal equation for the following polar curves.

① . $r^m = a^m \cos m\theta + b^m \sin m\theta$

Given.

$$r^m = a^m \cos m\theta + b^m \sin m\theta \longrightarrow ①$$

differentiate ① w.r.t θ

$$m r^{m-1} \frac{dr}{d\theta} = -\sin m\theta \cdot m a^m + \cos m\theta \cdot m b^m$$

$$\frac{r^m}{r} \frac{dr}{d\theta} = b^m \cos m\theta - a^m \sin m\theta$$

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{b^m \cos m\theta - a^m \sin m\theta}{a^m \cos m\theta + b^m \sin m\theta}$$

$$\cot \phi = \frac{b^m \cos m\theta - a^m \sin m\theta}{a^m \cos m\theta + b^m \sin m\theta}$$

$$1 + \cot^2 \phi = 1 + \frac{(b^m \cos m\theta - a^m \sin m\theta)^2}{(a^m \cos m\theta + b^m \sin m\theta)^2}$$

$$= \frac{(a^m \cos m\theta + b^m \sin m\theta)^2 + (b^m \cos m\theta - a^m \sin m\theta)^2}{(a^m \cos m\theta + b^m \sin m\theta)^2}$$

$$1 + \cot^2 \phi = \frac{2(a^{2m} + b^{2m})}{r^{2m}}$$

∴ W.K.T the pedal equation

$$\frac{1}{p^2} = \frac{1}{r^2}(1 + \cot^2 \phi)$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left(\frac{a^{2m} + b^{2m}}{r^{2m}}\right)$$

$$\Rightarrow \frac{1}{p^2} = \frac{a^{2m} + b^{2m}}{r^{2+2m}}$$

$$\boxed{\Rightarrow r^{2m+2} = p^2(a^{2m} + b^{2m})}$$

Module - II

Differential Calculus - II

I **Taylor's and Maclaurin's series expansions**

- The series expansions of a function $y=f(x)$ about a point $x=a$ is given by $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$ → ①

is called a taylor's series expansions.

- If $a=0$ then eq ① becomes $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$ is called a maclaurin's series expansion.

a. Obtain the taylor's series of expansion of $\log(\cos x)$ about a point $a = \pi/4$

$\Rightarrow f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots \longrightarrow ①$

$f(x) = \log(\cos x) \Rightarrow f(\pi/4) = \log(\cos \pi/4) = \log(1/\sqrt{2}) = -\log\sqrt{2}$

$f'(x) = \frac{(-\sin x)}{\cos x} \Rightarrow f'(x) = -\tan x \Rightarrow f(\pi/4) = -\tan \pi/4 = -1$

$f''(x) = -\sec^2 x \Rightarrow f''(\pi/4) = -\sec^2 \pi/4 = -2$

$f'''(x) = -2\sec^2 x \tan x \Rightarrow f'''(\pi/4) = -2\sec^2 \pi/4 . \tan \pi/4 = -4.$

sub in (1)

$\log(\cos x) = -\log\sqrt{2} + (x-\pi/4)(-1) + \frac{(x-\pi/4)^2}{2}(-2) + \frac{(x-\pi/4)^3}{6}(-4) + \cdots$

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Q.2 Obtain the ~~Taylor's~~ Power series expansion of $f(x) = \log x$ about a point of $x=1$. by considering the term up to 4th degrees.

w.k.t

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \frac{(x-a)^4}{4!} f^{iv}(a) + \cdots$$

$f(x) = \log x \Rightarrow f(1) = \log(1) = 0$

$f'(x) = 1/x \Rightarrow f'(1) = 1/1 = 1$

$f''(x) = -1/x^2 \Rightarrow f''(1) = -1/1^2 = -1$

$f'''(x) = 2/x^3 \Rightarrow f'''(1) = 2/1^3 = 2$

$f^{iv}(x) = -6/x^4 \Rightarrow f^{iv}(1) = -6/1 = -6$ //.

sub in eq^n (1) we get.

$$f(x) = 0 + (x-1)(1) + \frac{(x-1)^2}{2}(-1) + \frac{(x-1)^3}{6_3}(2) + \frac{(x-1)^4}{24_4}(-6) + \cdots$$

$$\log x = x - 1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \frac{(x-1)^4}{4} + \cdots //$$

Q.3 Find the series expansion of $\tan^{-1} x$ in powers of $(x-1)$.

Given $f(x) = \tan^{-1} x$ and $a = 1$

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3} f'''(a) + \cdots$$

$f(x) = \tan^{-1} x \Rightarrow f(1) = \tan^{-1}(1) = \pi/4$.

$f'(x) = \frac{1}{1+x^2} \Rightarrow f'(1) = \frac{1}{1+1^2} = \frac{1}{2}$.

$f''(x) = \frac{-1}{(1+x^2)^2} \times 2x = \frac{-2x}{(1+x^2)^2} \Rightarrow f''(1) = \frac{-2}{(1+1^2)^2} = -1/2$ //.

$$f'''(x) = \frac{(1+x^2)^2(-2) - (-2x) \times 2(1+x^2)(2x)}{(1+x^2)^4}.$$

$$f'''(1) = \frac{-8+16}{16} = 8/16 = 1/2$$

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sub in eq^n (1)

$f(x) = \pi/4 + (x-1)(1/2) + \frac{(x-1)^2}{2} \times (-1/2) + \frac{(x-1)^3}{6}(1/2) + \cdots$

$\tan^{-1}x = \pi/4 + \left(\frac{x-1}{2}\right) - \left(\frac{x-1}{2}\right)^2 + \left(\frac{(x-1)^3}{12}\right) + \cdots$ //.

* **Maclaurin's series**

1. Obtain the maclaurin's series of expression of $f(x) = \log(1+e^x)$.

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$

we have $f(x) = \log(1+e^x) = f(0) = \log(1+e^0) = \log(1+1) = \log 2$.

$f'(x) = \frac{e^x}{1+e^x} \Rightarrow f'(0) = \frac{e^0}{1+e^0} = \frac{1}{(1+1)} = 1/2$.

$f''(x) = \frac{(1+e^x)e^x - e^x(e^x)}{(1+e^x)^2} = \frac{e^x + e^{2x} - e^{2x}}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2}$.

$= f''(0) = \frac{e^0}{(1+e^0)^2} = \frac{1}{(1+1)^2} = 1/4$.

$f'''(x) = \frac{(1+e^x)^2(e^x) - e^x \cdot 2(1+e^x)e^x}{(1+e^x)^4} = f'''(0) = \frac{4-4}{24} = 0$ //.

sub in (1).

$f(x) = \log 2 + x/2 + \frac{x^2}{2}(1/4) + \frac{x^3}{3!}(0) + \cdots$

$\log(1+e^x) = \log 2 + x/2 + \frac{x^2}{8} + \cdots$ //

2. Find the power series expansion of $f(x) = \log(\sec x)$ in powers of x upto the terms containing x^4.

w.k.t.

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) \cdots$

we have $f(x) = \log(\sec x) = f(0) = \log(\sec 0) = \log 1 = 0$

$f'(x) = \frac{1}{\sec x} \sec x \tan x$ $\tan x$ $f'(0) = \tan 0 = 0$.

$f''(x) = \sec^2 x \Rightarrow f''(0) = \sec^2(0) = 1$

$f'''(x) = 2\sec x \cdot \sec x \tan x = 2\sec^2 x \tan x = f'''(0) = 2(\sec^2(0) \cdot \tan(0)) = 0$

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$f^{iv}(x) = 2\sec^2 x(\sec^2 x) + (2\tan x)(2\sec^2 x \cdot \tan x)$.

$f^{iv}(0) = 2\sec^2(0)(\sec^2(0)) + 0 = 2$ //.

sub in eq^n ①.

$f(x) = 0 + x \times 0 + \frac{x^2(1)}{2!} + \frac{x^3}{3!}(0) + \frac{x^2}{4!}(2) + \cdots$

$= \frac{x^2}{2} + \frac{x^4}{12} + \cdots$ //.

03. obtain the Maclaurin's series of expansion $f(x) = \tan^{-1}x$

Given $f(x) = \tan^{-1}x$.

w.k.t

$f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \cdots$

wehave.

$f(x) = \tan^{-1}x \Rightarrow f(0) = \tan^{-1}(0) = 0$.

$f'(x) = \frac{1}{1+x^2} = f'(0) = \frac{1}{1+0^2} = 1$

$f''(x) = \frac{-1}{(1+x^2)^2} \times 2x = \frac{-2x}{(1+x^2)^2} = f(0) = \frac{(-2\times 0)}{(1+0^2)^2} = 0$.

$f'''(x) = \frac{(1+x^2)^2(-2) - (-2x)\times 2(1+x^2)(2x)}{(1+x^2)^4}$.

$f'''(0) = \frac{-2+0}{(1+0)^4} = -2$ //

sub in eq ①

$f(x) = 0 + x \times 1 + \frac{x^2}{2!} \times 0 + \frac{x^3}{3!} \times -2 + \cdots$

$\tan^{-1}x = x - \frac{x^3}{3} + \cdots$ //.

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(04) obtain the Maclurin's series expansion of $f(x) = \log(1+\cos x)$. in power of x upto the terms containg x^4.

w.k.t.

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \cdots$$

we have

$$f(x) = \log(1+\cos x) \Rightarrow f(0) = \log(1+\cos(0)) = \log 2.$$

$$f'(x) = \frac{-\sin x}{1+\cos x} = \frac{-2\sin x/2 \cos x/2}{2\cos^2 x/2} = f'(x) = -\tan x/2$$

$$f'(0) = 0$$

$$f''(x) = -1/2 \sec^2 x/2 = f''(0) = -1/2 \sec^2(0) = -1/2.$$

$$f'''(x) = -1/2 \times 2\sec^2 x/2 \tan x/2 = f'''(0) = -\sec^2(0)\tan(0) = 0$$

$$f^{iv}(x) = -\sec^2 x/2 \cdot \sec^2 x/2 \times 1/2 - \tan x/2 \times 2\sec^2 x/2 \tan x/2$$

$$f^{iv}(0) = -1/2 \cdot \sec^4(0) - 0 = -1/2$$

sub in ①

$$f(x) = \log 2 + x \times 0 + \frac{x^2}{2} \times (-1/2) + \frac{x^3}{6} \times 0 + \frac{x^4}{24} \times \left(\frac{-1}{2}\right) + \cdots$$

$$\log(1+\cos x) = \log 2 - \frac{x^2}{4} - \frac{x^4}{48} + \cdots$$

*** 05. Find the maclurin's series of $\sqrt{1+\sin 2x}$ by considering the terms up to 4th degree.

w.k.t.

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \cdots \rightarrow ①$$

we have.

$$f(x) = \sqrt{1+\sin 2x}$$

$$= \sqrt{\sin^2 x + \cos^2 x + 2\sin x \cos x} = \sqrt{(\sin x + \cos x)^2}$$

$$f(x) = \sin x + \cos x \Rightarrow f(0) = 0+1 = 1.$$

$$f'(x) = \cos x - \sin x \Rightarrow f(0) = 1-0 = 1$$

$$f''(x) = -\sin x - \cos x \Rightarrow f(0) = 0-1 = -1$$

$$f'''(x) = -\cos x + \sin x \Rightarrow f(0) = -1+0 = -1$$

$$f^{iv}(x) = \sin x + \cos x \Rightarrow f(0) = 0+1 = 1$$

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sub in (1) we get.

$$f(x) = 1 + x \times 1 + \frac{x^2}{2!} \times -1 + \frac{x^3}{3!} + -4\frac{x^4}{4!} \times 1 + \cdots$$

$$\sqrt{1+\sin 2x} = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

06. Q. obtain the series expansion of $e^{\sin x}$ in power of x.

Given

$f(x) = e^{\sin x}$.

w.k.t.

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots \rightarrow (1).$$

we have.

$f(x) = e^{\sin x}$ $\quad f(0) = e^{\sin(0)} = e^0 = 1.$

$f'(x) = e^{\sin x} \cdot \cos x = f'(x) = f(x)\cos x = f'(0) = f(0) \times \cos(0)$

$= 1$ //.

$f''(x) = f(x) \cdot -\sin x + \cos x \cdot f'(x) = f''(0) = -f(0) \cdot \sin(0) + \cos(0) \cdot f'(0)$

$1 - 0 + 1 \cdot 1$

$= 0 + 1$

$= 1$

$f'''(x) = -f(x)\cos x - \sin x\, f'(x) - \sin x\, f'(x) + \cos x \cdot f''(x).$

$f'''(0) = -f(0)\cos(0) - \sin(0) f'(0) - \sin(0) f'(0) + \cos(0) \cdot f''(0)$

$= -1 - 0 - 0 + 1$

$= 0$

Sub in (1) we get.

$$f(x) = 1 + x \times 1 + \frac{x^2}{2!} \times 1 + \frac{x^3}{3!} \times 0 + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \cdots \text{ //.}$$

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* In Determinate forms :-

If the expansion $f(x)$ at $x = a$ assumes the forms like. $0/0$, ∞/∞, $0\times\infty$, 0^0, ∞^0, 1^∞ etc. which do not reperesents any value are called the indeterminate forms.

$+\infty - \infty$

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* L Hospital's rule

If $f(x)$ and $g(x)$ are any two functions such that

(i) $\lim_{x\to a} f(x) = f(a) = 0$

$\lim_{x\to 0} g(x) = g(a) = 0$

(ii) $f'(x)$ and $g'(x)$ exists then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

(iii) further if $f'(a) = 0 = g'(a)$ then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f''(x)}{g''(x)}$$

Q.1 Type one

$0/0$ and ∞/∞ forms

1. Evalute the following.

(i) $\lim_{x\to 0} \frac{xe^x - \log(1+x)}{x^2}$

(ii) $\lim_{x\to 0} \frac{1-\cos x}{x\log(1+x)}$

(iii) $\lim_{x\to \pi/2} \frac{\log(\sin x)}{(\pi/2 - x)^2}$

(iv) $\lim_{x\to 0} \frac{a^x - b^x}{x}$

(v) $\lim_{x\to \pi/4} \frac{\sec^2 x - 2\tan x}{1+\cos 4x}$

(vi) $\lim_{x\to \pi/2} \frac{\log(\cos x)}{\tan x}$.

(vii) $\lim_{x\to \pi/2} \frac{\log(x - \pi/2)}{\tan x}$.

(viii) $\lim_{x\to 0} \log_{\sin x} \sin 2x$.

(ix) $\lim_{x\to 0} \log_{\tan bx} \tan ax$

(x) $\lim_{x\to 0} \frac{\log x}{\operatorname{cosec} x}$.

Q1. let $L = \lim_{x \to 0} \frac{xe^x - \log(1+x)}{x^2}$ $\left(\frac{0}{0}\right)$

Appling LHR, we get

$$L = \lim_{x \to 0} \frac{xe^x + e^x - \left(\frac{1}{1+x}\right)}{2x} = \left(\frac{0}{0}\right)$$

using LHR, we get

$$L = \lim_{x \to 0} \frac{xe^x + e^x + e^x + \frac{1}{(1+x^2)}}{2}$$

$$= \frac{0+1+1+1}{2} = 3/2.$$

Q2. let $L = \lim_{x \to 0} \frac{1-\cos x}{x \log(1+x)}$ $\left(\frac{0}{0}\right)$

using LHR, we get

$$L = \lim_{x \to 0} \frac{\sin x}{\frac{x}{(1+x)} + \log(1+x)} = \left(\frac{0}{0}\right)$$

using LHR, we get

$$L = \lim_{x \to 0} = \frac{\cos x}{\frac{1+x-x}{(1+x)^2} + \frac{1}{1+x}}$$

$$= \lim_{x \to 0} \frac{\cos x}{\frac{1}{(1+x)^2} + \frac{1}{(1+x)}}$$

$$L = 1/1+1 = 1/2 //.$$

Q3. let $L = \lim_{x \to \pi/2} \frac{\log(\sin x)}{(\pi/2 - x)^2}$ $\left(\frac{0}{0}\right)$

using LHR we get

$$L = \lim_{x \to \pi/2} \frac{\frac{1}{\sin x} \times \cos x}{2(\pi/2 - x)(-1)}$$

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$$L = -\frac{1}{2} \lim_{x \to \pi/2} \frac{\cot x}{\pi/2 - x} \left(\frac{0}{0}\right)$$

using LHR, we get

$$L = \lim_{x \to 0} \frac{\cos x}{\frac{1+x-x}{(1+x)^2} + \frac{1}{1+x}}.$$

$$= \lim_{x \to 0} \frac{\cos x}{\frac{1}{(1+x)^2} + \frac{1}{(1+x)}}$$

$$L = \frac{1}{1+1} = 1/2.$$

03. let $L = \lim_{x \to \pi/2} \frac{\log(\sin x)}{(\pi/2 - x)^2} \left(\frac{0}{0}\right)$

using LHR we get

$$L = \lim_{x \to \pi/2} \frac{\frac{1}{\sin x} \times \cos x}{2(\pi/2 - x)(-1)}$$

$$L = -1/2 \lim_{x \to \pi/2} \frac{\cot x}{\pi/2 - x} \left(\frac{0}{0}\right)$$

using LHR we get

$$L = -1/2 \lim_{x \to \pi/2} \frac{\cancel{-}\operatorname{cosec}^2 x}{\cancel{-}1}$$

$$= -1/2 \times \operatorname{cosec}^2 \pi/2 = -1/2 \times 1 = -1/2$$

04. let $L = \lim_{x \to 0} \frac{a^x - b^x}{x} \left(\frac{0}{0}\right)$

using LHR we get

$$L = \lim_{x \to 0} = \frac{a^x \log a - b^x \log b}{1}$$

$$= a^0 \log a - b^0 \log b$$

$$\log a - \log b = \log(a/b)$$

5. $\lim_{x\to\pi/4} \frac{\sec^2 x - 2\tan x}{1+\cos 4x}$.

$\frac{\sec^2 \pi/4 - 2\tan \pi/4}{1+\cos 4(\pi/4)} = \left(\frac{0}{0}\right)$

using LHR, we get.

$\frac{2\sec^2 x \cdot \tan x - 2\sec^2 x}{0 - \sin 4x \cdot 4}$

~~$\lim_{x\to\pi/4} = \frac{2\sec^2 \pi/4 \cdot \tan \pi/4 - 2\sec^2 \pi/4}{0 - \sin 4\pi/4 \times 4}$~~

$\lim_{x\to\pi/4} \frac{\sec^2 x \tan x - \sec^2 x}{-2\sin 4x} = \left(\frac{0}{0}\right)$.

using LHR we get

$L = \lim_{x\to\pi/4} \frac{\sec^4 x + (\tan x)(2\sec^2 x \tan x) - (2\sec^2 x \tan x)}{-2\cos 4x \times 4}$.

$= \frac{\sec^4 \pi/4 + (\tan \pi/4)(2\sec^2 \pi/4 \tan \pi/4) - (2\sec^2 \pi/4 \tan \pi/4)}{-2\cos 4(\pi/4) \times 4}$.

$= \frac{4+4-4}{-8(-1)} = \frac{4}{8} = 1/2$ //.

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6 let $L = \lim_{x\to\pi/2} \frac{\log(\cos x)}{\tan x} \left(\frac{\infty}{\infty}\right)$

Applying LHR.

$L = \lim_{x\to\pi/2} \frac{\frac{1}{\cos x} \times -\sin x}{\sec^2 x} = \lim_{x\to\pi/2} \frac{-\tan x}{\sec^2 x} = \left(\frac{\infty}{\infty}\right)$

$L = -\lim_{x\to\pi/2} \frac{\cos^2 x}{\cot x} \left(\frac{0}{0}\right)$

Applying LHR.

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$$L = -\lim_{x\to\pi/2} \frac{\cancel{+}2\cos x \sin x}{\cancel{+}\cancel{2}\operatorname{cosec}^2 x}$$

$$= -\frac{2\cos\pi/2 \cdot \sin\pi/2}{\operatorname{cosec}^2 \pi/2} = \frac{2\times 0\times 1}{1} = 0$$

7. let $L = \lim_{x\to\pi/2} \frac{\log(x-\pi/2)}{\tan x} \left(\frac{\infty}{\infty}\right)$

Applying LHR

$$L = \lim_{x\to\pi/2} \frac{\frac{1}{x-\pi/2}}{\frac{\sec^2 x}{1}} \left(\frac{\infty}{\infty}\right)$$

$$L = \lim_{x\to\pi/2} \frac{1}{x-\pi/2} \times \frac{1}{\sec^2 x}$$

$$L = \lim_{x\to\pi/2} \frac{\cos^2 x}{x-\pi/2} \left(\frac{0}{0}\right)$$

$$L = \lim_{x\to\pi/2} -\frac{2\cos x \sin x}{1}$$

$$= -2\cos\pi/2 \cdot \sin\pi/2 = 0 //$$

8. let $L = \lim_{x\to 0} \log_{\sin 2x} \sin 2x = \lim_{x\to 0} \frac{\log \sin 2x}{\log \sin x} \left(\frac{\infty}{\infty}\right)$

Applying L'HR we get

$$L = \lim_{x\to 0} \frac{\frac{1}{\sin 2x} \times \cos 2x \times 2}{\frac{1}{\sin x} \times \cos x} = 2\lim_{x\to 0} \frac{\cot 2x}{\cot x} \left(\frac{\infty}{\infty}\right)$$

$$L = 2\lim_{x\to 0} \frac{\tan x}{\tan 2x} \left(\frac{0}{0}\right)$$

using LHR

$$L = \cancel{2}\lim_{x\to 0} \frac{\sec^2 x}{\sec^2 2x \times \cancel{2}} = \frac{1}{1} = 1$$

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9. let

$$L = \lim_{x\to 0} \log_{\tan bx} \tan ax = \lim_{x\to 0} \frac{\log \tan ax}{\log \tan bx}$$

Applying LHR,

$$L = \lim_{x\to 0} \frac{\frac{1}{\tan ax} \times \sec^2 ax \times a}{\frac{1}{\tan bx} \times \sec^2 bx \times b}$$

$$\frac{a}{b} \lim_{x\to 0} \frac{\frac{\cos ax}{\sin bx} \times \frac{1}{\cos^2 ax}}{\frac{\cos bx}{\sin bx} \times \frac{1}{\cos^2 bx}}$$

$$\frac{a}{b} \lim_{x\to 0} \frac{2 \sin bx \cos bx}{2 \sin ax \cos ax} = \frac{a}{b} \lim_{x\to 0} \frac{\sin 2bx}{\sin 2ax} \left(\frac{0}{0}\right)$$

using LHR.

$$L = \frac{a}{b} \lim_{x\to 0} \frac{\cos 2bx \times 2b}{\cos 2ax \times 2a} = 1$$

10. let

$$L = \lim_{x\to 0} \frac{\log x}{\operatorname{cosec} x} \left(\frac{\infty}{\infty}\right)$$

Applying LHR.

$$L = \lim_{x\to 0} \frac{1/x}{-\operatorname{cosec} x \cdot \cot x} = -\lim_{x\to 0} \frac{\sin x \times \tan x}{x} \left(\frac{0}{0}\right)$$

$$= -\lim_{x\to 0} \left(\frac{\sin x}{x}\right) \times \lim_{x\to 0} (\tan x)$$

$$= -1 \times \tan(0) = -1 \times 0$$

$= 0$ //

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Type 2 $0^0, \infty^0, 1^\infty$ forms.

Evalute the following:

01. $\lim_{x\to 1} x^{\frac{1}{1-x}}$

02. $\lim_{x\to \pi/2} (\sin x)^{\tan x}$.

03. $\lim_{x\to 1} (1-x^2)^{\frac{1}{\log(1-x)}}$

04. $\lim_{x\to 0} \left(2-\frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$

05. $\lim_{x\to 0} \left(\frac{a^x+b^x+c^x}{3}\right)^{1/x}$.

06. $\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x}$

07. $\lim_{x\to \infty} \left(\frac{\pi}{2} - \tan^{-1}x\right)^{1/x}$

08. $\lim_{x\to 0} (\cot x)^{\tan x}$

09. $\lim_{x\to 0} x^{\sin x}$.

01. Let $L = \lim_{x\to 1} x^{\frac{1}{1-x}}$ (1^∞)

$\log L = \lim_{x\to 1} \log\left\{x^{\frac{1}{1-x}}\right\}$

$= \lim_{x\to 1} \frac{1}{1-x} \cdot \log x$.

$\log L = \lim_{x\to 1} \frac{\log x}{1-x} \left(\frac{0}{0}\right)$

Applying LHR.

$\log L = \lim_{x\to 1} \frac{1/x}{-1}$

$\log L = -1$

$L = e^{-1}$

02. Let $L = \lim_{x\to \pi/2} (\sin x)^{\tan x} (1^\infty)$

$\log L = \lim_{x\to \pi/2} \log(\sin x)^{\tan x}$

$= \lim_{x\to \pi/2} \tan x \cdot \log(\sin x)$

$\lim_{x\to \pi/2} \frac{\log(\sin x)}{\cot x} \left(\frac{0}{0}\right)$

using LHR

$\log L = \lim_{x\to \pi/2} \frac{\frac{1}{\sin x} \times \cos x}{-\operatorname{cosec}^2 x}$

$= -\lim_{x\to \pi/2} \frac{\cot x}{\operatorname{cosec}^2 x}$

$\log L = \frac{-0}{1} = 0$

$L = e^0 = 1$

03. Let $L = \lim_{x\to 1} (1-x^2)^{\frac{1}{\log(1-x)}}$ (0^0)

$\log L = \lim_{x\to 1} \log(1-x^2)^{\frac{1}{\log}}$

$= \lim_{x\to 1} \frac{1}{\log(1-x)} \cdot \log(1-x^2)$

$\log L = \lim_{x\to 1} \frac{\log(1-x^2)}{\log(1-x)} \left(\frac{\infty}{\infty}\right)$

applying LHR.

$$\log L = \lim_{x\to 1} \frac{\frac{1}{(1-x^2)} \times (-2x)}{\frac{1}{(1-x)} \times (-1)}$$

$$= 2 \lim_{x\to 1} \frac{x}{(1-x)(1+x)} \times (1-x)$$

$$\log L = 2 \times \frac{1}{(1+1)} = 2 \times \frac{1}{2} = 1$$

$L = e.$

4. let $L = \lim_{x\to a} \left(2 - x/a\right)^{\tan\left(\frac{\pi x}{2a}\right)}$ (1^∞)

$$\log L = \lim_{x\to a} \log\left(2 - x/a\right)$$

$$= \lim_{x\to a} \tan\left(\frac{\pi x}{2a}\right) \cdot \log\left(2 - \frac{x}{a}\right)$$

$$\log L = \lim_{x\to a} \frac{\log\left(2 - x/a\right)}{\cot\left(\frac{\pi x}{2a}\right)} \quad \left(\frac{0}{0}\right)$$

Applying LHR, we get.

$$\log L = \lim_{x\to a} \frac{\frac{1}{(2 - x/a)} \times (-1/a)}{-\operatorname{cosec}^2(\pi x/2a) \times \pi/2a}$$

$$= 2/\pi \times \frac{1}{\operatorname{cosec}^2(\pi/2)} = 2/\pi \times 1$$

$$\log L = 2/\pi \Rightarrow L = e^{2/\pi}$$

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05. let

$$L = \lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{1/x} \quad (1^\infty)$$

$$\log L = \lim_{x\to 0} \log\left(\frac{a^x+b^x+c^x}{3}\right)^{1/x} = \lim_{x\to 0} \frac{\log\left(\frac{a^x+b^x+c^x}{3}\right)}{x} = \left(\frac{0}{0}\right)$$

applying LHR we get.

$$\log L = \lim_{x\to 0} \frac{\frac{1}{\left(\frac{a^x+b^x+c^x}{3}\right)} \times \frac{1}{3}\left\{a^x\log a + b^x\log b + c^x\log c\right\}}{1}$$

$$1/3\left\{\frac{1}{\left(\frac{a^0+b^0+c^0}{3}\right)} \times \left[a^0\log a + b^0\log b + c^0\log c\right]\right\}.$$

$$1/3\left\{1/(3/3)\left\{\log a + \log b + \log c\right\}\right\}$$

$$= 1/3\log(abc).$$

$$\log L = \log(abc)^{1/3} =$$

$$L = (abc)^{1/3}$$

(6) let

$$L = \lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2} (1^\infty)$$

$$\log L = \lim_{x\to 0} \log\left(\frac{\tan x}{x}\right)^{1/x}.$$

$$\lim_{x\to 0} \frac{\log\left(\frac{\tan x}{x}\right)}{x} \left(\frac{0}{0}\right)$$

applying LHR

$$\log L = \lim_{x\to 0} \frac{1}{\left(\frac{\tan x}{x}\right)} \left\{\frac{x(\sec^2 x) - \tan x}{x^2}\right\}$$

$$= \lim_{x\to 0} \frac{x\sec^2 x - \tan x}{x^2} \left(\frac{0}{0}\right)$$

LHR

$$\log L = \lim_{x\to 0}\left\{\frac{\sec^2 x + x(2\sec^2 x\tan x) - \sec^2 x}{2x}\right\}$$

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$$\log L = \lim_{x\to 0} \frac{2x \sec^2 x \tan x}{2x}$$

$$= \lim_{x\to 0} \sec^2 x \tan x.$$

$$= \sec^2(0) \times \tan(0)$$

$$= 1 \times 0$$

$$\log L = 0$$

$$L = e^0 = 1$$

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let

$$L = \lim_{x\to\infty} \left(\pi/2 - \tan^{-1} x\right)^{1/x} = (0^0)$$

$$\log L = \lim_{x\to\infty} \log\left[\left(\pi/2 - \tan^{-1} x\right)^{1/x}\right]$$

$$= \lim_{x\to\infty} \frac{\log\left(\pi/2 - \tan^{-1} x\right)}{x} \left(\frac{\infty}{\infty}\right)$$

applying LHR, we get

$$\log L = \lim_{x\to\infty} \frac{\frac{1}{(\pi/2 - \tan^{-1} x)} \left\{0 - \frac{1}{(1+x^2)}\right\}}{1}$$

$$= -\lim \frac{1/(1+x^2)}{\pi/2 - \tan^{-1} x} \left(\frac{0}{0}\right)$$

using LHR.

$$\log L = -\lim_{x\to\infty} \frac{\cancel{-}\frac{1}{(1+x^2)^2} \times 2x}{\cancel{-}\frac{1}{\cancel{(1+x^2)}}}$$

$$= -\lim_{x\to\infty} \frac{2x}{1+x^2} \left(\frac{\infty}{\infty}\right)$$

using LHR

$$\log L = -\lim_{x\to\infty} \frac{2}{2x}$$

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$$= \frac{-1}{\infty} = 0$$

$$L = e^0 = 1$$

8\. let $L = \lim_{x \to 0} (\cot x)^{\tan x}$ (∞^0)

$$\log L = \lim_{x \to 0} \log (\cot x)^{\tan x}.$$

$$= \lim_{x \to 0} \tan x \times \log (\cot x)$$

$$= \lim_{x \to 0} \frac{\log (\cot x)}{\cot x} \left(\frac{\infty}{\infty}\right)$$

applying LHR

$$\log L = \lim_{x \to 0} \frac{\frac{1}{\cot x} \times -\cancel{\csc^2 x}}{-\cancel{\csc^2 x}}$$

$$= \lim_{x \to 0} \tan x = 0$$

$$L = e^0 = 1.$$

9\. let $L = \lim_{x \to 0} x^{\sin x}$ (0^0)

$$\log L = \lim_{x \to 0} \log (x^{\sin x})$$

$$= \lim_{x \to 0} \sin x \times \log x$$

$$= \lim_{x \to 0} \frac{\log x}{\csc x} \left(\frac{\infty}{\infty}\right)$$

applying LHR

$$\log L = \lim_{x \to 0} \frac{1/x}{-\csc x \cdot \cot x}.$$

$$= -\lim_{x \to 0} \frac{\sin x \times \tan x}{x}.$$

$$\log L = -\lim_{x \to 0} \left(\frac{\sin x}{x}\right) \times \tan x$$

$= -1 \times \tan(0) = -1 \times 0$

$\log L = 0 = L = e^0 = 1$ //.

* Partial differentiation.

Partial derivatives :-

- If $u = f(x, y)$ is function of two independent variable x and y then the partial derivative of u with respect to x is defined as $\frac{\partial u}{\partial x}$ (or) $U_x = \lim_{x \to 0} \frac{f(x + \delta x, y) - f(x-y)}{\delta x}$.

(Diffⁿ u w.r.t x, assuming y as constant).

- Similarly a partial derivative of u w.r.t y is defined as $\frac{\partial u}{\partial y}$ (or) $U_y = \lim_{\delta y \to 0} \frac{f(x, y + \delta y) - f(x, y)}{\delta y}$.

(Diffⁿ u w.r.t y, assuming x as constant).

* Higher order partial derivatives

$u = f(x, y)$

- $\frac{\partial u}{\partial x}$ or U_x
  - $\frac{\partial^2 u}{\partial x^2}$ or U_{xx}
  - $\frac{\partial^2 u}{\partial x \partial y}$ or U_{xy}
- $\frac{\partial u}{\partial y}$ or U_y
  - $\frac{\partial^2 u}{\partial y \partial x}$ (or) U_{yx}
  - $\frac{\partial^2 u}{\partial y^2}$ or U_{yy} (Second order)

(U_{xy}, U_{yx}: mixed partial derivatives)

Note $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ (or) $U_{xy} = U_{yx}$

are always equal.

03/10/2014

Q.1 if $q = u = x^3 - 3xy^2 + x + e^x (\cos y + 1) \to 1$ then proove that
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Given $u = x^3 - 3xy^2 + x + e^x \cos y + 1 \to (1)$

Diffⁿ eqⁿ (1) partially w.r.t x.

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2(1) + 1 + (\cos y)e^x$$

$$\frac{\partial^2 u}{\partial x^2} = 6x + (\cos y)e^x \to (2)$$

Diffⁿ eqⁿ (1) partially w.r.t y

$$\frac{\partial u}{\partial y} = -3x(2y) + e^x(-\sin y)$$

$$\frac{\partial^2 u}{\partial y^2} = -6x - e^x \cos y \to (3)$$

(2) + (3) =

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x + e^x \cos y - 6x - e^x \cos y$$
$$= 0$$

**

Q.2. If $u = e^{-2\pi^2 t} \sin \pi x \sin \pi y$, then p.t $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$

Given

$$u = e^{-2\pi^2 t} \sin \pi x \sin \pi y \to (1)$$

Diffⁿ eqⁿ (1) partially w.r.t x.

$$\frac{\partial u}{\partial x} = e^{-2\pi^2 t} \sin \pi y \cdot \cos \pi x \times \pi$$

$$\frac{\partial^2 u}{\partial x^2} = e^{-2\pi^2 t} \sin \pi y \times \pi X - \sin \pi x \pi$$

$$\frac{\partial^2 u}{\partial x^2} = -\pi^2 e^{-2\pi^2 t} \sin \pi y \sin \pi x \to (2)$$

|||ly

$$\frac{\partial^2 u}{\partial y^2} = -\pi^2 e^{-2\pi^2 t} \sin \pi x \sin \pi y \to (3)$$

Diffⁿ eqⁿ (1) paratially w.r.t t.

$$\frac{\partial u}{\partial t} = \sin\pi x \cdot \sin\pi y \times e^{-2\pi^2 t} \times (-2\pi^2)$$

$$\frac{\partial u}{\partial t} = -2\pi^{2} e^{-2\pi^2 t} \sin\pi x \sin\pi y \rightarrow (4)$$

(2) + (3) $\Rightarrow$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -2\pi^2 e^{-2\pi^2 t} \sin\pi x \sin\pi y.$$

using (4).

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$$

3. If $u = e^{ax+by} f(ax-by)$ then S.T $b\frac{\partial u}{\partial x} + a\frac{\partial u}{\partial y} = 2aby$.

$\Rightarrow$

$u = e^{ax+by} f(ax-by) \rightarrow (1)$

Diffⁿ eqⁿ (1) partially w.r.t x.

$$\frac{\partial u}{\partial x} = e^{ax+by} f'(ax-by)(a) + f(ax-by) \times e^{ax+by} \times a.$$

$$= a e^{ax+by} \{ f'(ax-by) + f(ax-by) \}.$$

$$b\frac{\partial u}{\partial x} = ab e^{ax+by} \{ f'(ax-by) + f(ax-by) \} \rightarrow (2)$$

Diffⁿ eqⁿ (1) partially w.r.t y.

$$\frac{\partial u}{\partial y} = e^{ax+by} f'(ax-by)(-b) + f(ax-by) e^{ax+by} \times b$$

$$= b e^{ax+by} \{ -f'(ax-by) + f(ax-by) \}$$

$$a\frac{\partial u}{\partial y} = ab e^{ax+by} \{ -f'(ax-by) + f(ax-by) \} \rightarrow (3)$$

(2)+(3) $\Rightarrow$

$$b\frac{\partial u}{\partial x} + a\frac{\partial u}{\partial y} = abe^{ax+by}\{2f(ax-by)\} = 2abu$$

04/10/2017

4. If $v = e^{a\theta}\cos(a\log r)$ then P.T $\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0$.

$v = e^{a\theta}\cos(a\log r) \rightarrow ①$

Diffⁿ eqⁿ ① partially w.r.t r

$$\frac{\partial v}{\partial r} = e^{a\theta} \times -\sin(a\log r) \times a \times \frac{1}{r}$$

$$\frac{\partial v}{\partial r} = -ae^{a\theta} \cdot \frac{\sin(a\log r)}{r}$$

Diffⁿ again w.r.t r

$$\frac{\partial^2 v}{\partial r^2} = -ae^{a\theta}\frac{\left\{r \cdot \cos(a\log r) \times \frac{a}{r} - \sin(a\log r) \times 1\right\}}{r^2}$$

$$\frac{\partial^2 v}{\partial r^2} = -a^2e^{a\theta}\frac{\cos(a\log r)}{r^2} + ae^{a\theta}\frac{\sin(a\log r)}{r^2}$$

Diffⁿ eqⁿ ① partially w.r.t θ.

$$\frac{\partial v}{\partial \theta} = \cos(a\log r) \times e^{a\theta} \times a$$

$$\frac{\partial^2 v}{\partial \theta^2} = \cos(a\log r) \times e^{a\theta} \times a^2$$

Consider

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = \frac{-a^2e^{a\theta}\cos(a\log r)}{r^2} + \frac{ae^{a\theta}\sin(a\log r)}{r^2}$$

$$- \frac{ae^{a\theta}\sin(a\log r)}{r^2} + \frac{a^2e^{a\theta}\cos(a\log r)}{r^2}$$

$$= 0$$

* Note

Suppose $u = f(x, y, z)$ is a symmetric function then by finding any one partial derivative we write the other partial derivative directly.

5. If $u = \log\sqrt{x^2+y^2+z^2}$ then prove that $(x^2+y^2+z^2)(u_{xx}+u_{yy}+u_{zz}) = 1$

$u = \log\sqrt{x^2+y^2+z^2}$

$u = \frac{1}{2}\log(x^2+y^2+z^2) \rightarrow (1)$

Diffⁿ eqⁿ (1) partially w.r.t x

$$\frac{\partial u}{\partial x} = \frac{1}{2} \times \frac{1}{x^2+y^2+z^2} \times 2x = \frac{x}{x^2+y^2+z^2}$$

Diffⁿ again partially w.r.t x.

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2+z^2)(1) - x(2x)}{(x^2+y^2+z^2)^2}$$

$$u_{xx} = \frac{-x^2+y^2+z^2}{(x^2+y^2+z^2)^2}$$

|||ly $u_{yy} = \frac{-y^2+z^2+x^2}{(x^2+y^2+z^2)^2}$

$$u_{zz} = \frac{-z^2+x^2+y^2}{(x^2+y^2+z^2)^2}$$

$$u_{xx}+u_{yy}+u_{zz} = \frac{-x^2+y^2+z^2 - z^2+x^2+y^2 - y^2+z^2+x^2}{(x^2+y^2+z^2)^2}$$

$$\frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^2} = \frac{1}{(x^2+y^2+z^2)}$$

$$(x^2+y^2+z^2)(u_{xx}+u_{yy}+u_{zz}) = 1$$

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(6). If $u = \log(x^3+y^3+z^3-3xyz)$ then p.t

(a) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$

(b) $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$

⇒

Given $u = \log(x^3+y^3+z^3-3xyz) \rightarrow (1)$

Diffⁿ eqⁿ (1) partially w.r.t x.

$$\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz} \times (3x^2-3yz)$$

$$\frac{\partial u}{\partial x} = \frac{3(x^2-yz)}{(x^3+y^3+z^3-3xyz)}$$

$$\frac{\partial u}{\partial y} = \frac{3(y^2-xz)}{(x^3+y^3+z^3-3xyz)}$$

$$\frac{\partial u}{\partial z} = \frac{3(z^2-xy)}{x^3+y^3+z^3-3xyz}.$$

(a) consider.

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2-yz-xz-xy)}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3\cancel{(x^2+y^2+z^2-yz-xz-xy)}}{(x+y+z)\cancel{(x^2+y^2+z^2-xy-yz-zx)}}$$

$$= \frac{3}{(x+y+z)}.$$

(b) $$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= \frac{-3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2}$$

$$= \frac{-9}{(x+y+z)^2}$$ //

(07) If $u = \tan^{-1}(y/x)$ → then prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

$u = \tan^{-1}(y/x)$ → ①

Diffⁿ eqⁿ ① partially w.r.t x and y.

$$\frac{\partial u}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}} \times y \times \frac{-1}{x^2}$$

$$= \frac{1}{\left(\frac{x^2+y^2}{x^2}\right)} \times \frac{-y}{x^2}$$

$$\frac{\partial u}{\partial x} = \frac{-y}{x^2+y^2} \rightarrow ②$$

$$\frac{\partial u}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}} \times \frac{1}{x} \times 1$$

$$= \frac{1}{\left(\frac{x^2+y^2}{x^2}\right)} \times \frac{1}{x}$$

$$\frac{\partial u}{\partial y} = \frac{x}{x^2+y^2} \rightarrow ③$$

Diffⁿ eqⁿ ② partially w.r.t y.

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{(x^2+y^2)(-1)-(-y)(2y)}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \rightarrow ④$$

Diffⁿ eqⁿ ③ partially w.r.t x

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{(x^2+y^2)(1)-x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \rightarrow ⑤$$

from ④ and ⑤ we get

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$ //.

05/10/2017

*** 08 Verify that $u_{xy} = u_{yx}$ given that $u = x^y \rightarrow (1)$

Given $u = x^y \rightarrow (1)$

Diffⁿ eqⁿ (1) partially w.r.t x and y.

$$\frac{\partial u}{\partial x} = y\, x^{y-1} \rightarrow (2)$$

$$\frac{\partial u}{\partial y} = x^y \log x \rightarrow (3)$$

Diffⁿ eqⁿ (2) partially w.r.t y.

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = x^{y-1}(1) + y.x^{y-1} \times \log x.$$

$$\frac{\partial^2 u}{\partial y \partial x} \text{ (or) } u_{yx} = x^{y-1}\left[1 + y \log x\right] \rightarrow (4)$$

Diffⁿ eqⁿ (3) partially w.r.t x.

$$\frac{\partial^2 u}{\partial x \partial y} \text{ (or) } u_{xy} = x^{y-1}\left[1 + y \log x\right] \rightarrow (5)$$

from (4) and (5)

$= u_{xy} = u_{yx}$ //

* **Jacobian**

$$J\left(\frac{u, v}{x, y}\right) \text{ or } \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$J\left(\frac{u, v, w}{x, y, z}\right) \text{ or } \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Q.1 Find the Jacobiane of u, v, w with respect to x, y, z, Given that

$u = x+y+z$, $v = y+z$, $w = z$

$\frac{\partial u}{\partial x} = 1$ $\frac{\partial v}{\partial x} = 0$ $\frac{\partial w}{\partial x} = 0$

$\frac{\partial u}{\partial y} = 1$ $\frac{\partial v}{\partial y} = 1$ $\frac{\partial w}{\partial y} = 0$

$\frac{\partial u}{\partial z} = 1$ $\frac{\partial v}{\partial z} = 1$ $\frac{\partial w}{\partial z} = 1$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

(2) Find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ where $= x^2+y^2+z^2$, $v = xy+yz+xz$, $w = x+y+z$

$u = x^2+y^2+z^2$, $v = xy+yz+xz$, $w = x+y+z$.

$\frac{\partial u}{\partial x} = 2x$ $\frac{\partial v}{\partial x} = y+z$ $\frac{\partial w}{\partial x} = 1$

$\frac{\partial u}{\partial y} = 2y$ $\frac{\partial v}{\partial y} = x+z$ $\frac{\partial w}{\partial y} = 1$

$\frac{\partial u}{\partial z} = 2z$ $\frac{\partial v}{\partial z} = y+x.$ $\frac{\partial w}{\partial z} = 1$

$$\begin{vmatrix} 2x & 2y & 2z \\ y+z & x+z & y+x \\ 1 & 1 & 1 \end{vmatrix}$$

$2x(x+z - (y+x)) - 2y((y+z) - (y+x)) + 2z((y+z) - (x+z))$

$2x(x+z-y-x) - 2y(y+z-y-x) + 2z(y+z-x-z)$

~~$2x(z-y) - 2y(z-x) + 2z(y-x)$~~

$2xz - 2yx - 2yz + 2yx + 2zy - 2xz = 0$

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(3) If $u = yz/x$, $v = zx/y$, $w = \frac{xy}{z}$ then S.T. $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$.

$u = yz/x \quad v = zx/y \quad w = xy/z$

$\frac{\partial u}{\partial x} = \frac{-yz}{x^2} \quad \frac{\partial v}{\partial x} = z/y \quad \frac{\partial w}{\partial x} = y/z$

$\frac{\partial u}{\partial y} = z/x \quad \frac{\partial v}{\partial y} = \frac{-zx}{y^2} \quad \frac{\partial w}{\partial y} = x/z$

$\frac{\partial u}{\partial z} = y/x \quad \frac{\partial v}{\partial z} = x/y \quad \frac{\partial w}{\partial z} = -xy/z^2$

w.k.t

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} -\frac{yz}{x^2} & z/x & y/x \\ z/y & -zx/y^2 & x/y \\ y/z & x/z & -xy/z^2 \end{vmatrix}$$

$$= -yz/x^2\left[\frac{x^2}{yz} - \frac{x^2}{yz}\right] - z/x\left[-x/z - x/z\right] + y/x\left[x/y + x/y\right]$$

$$= 0 - z/x\left(\frac{-2x}{z}\right) + y/x\left(\frac{2x}{y}\right)$$

$$= 2+2 = 4 //.$$

*4 If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.
prove that $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$.

$x = r\sin\theta\cos\phi$	$y = r\sin\theta\sin\phi$	$z = r\cos\theta$
$\frac{\partial x}{\partial r} = \sin\theta\cos\phi$	$\frac{\partial y}{\partial r} = \sin\theta\sin\phi$	$\frac{\partial z}{\partial r} = \cos\theta$
$\frac{\partial x}{\partial \theta} = \cos\theta \cdot \cos\phi \cdot r$	$\frac{\partial y}{\partial \theta} = \cos\theta \cdot r \cdot \sin\phi$	$\frac{\partial z}{\partial \theta} = -\sin\theta \cdot r$
$\frac{\partial x}{\partial \phi} = -\sin\phi \cdot \sin\theta \cdot r$	$\frac{\partial y}{\partial \phi} = \cos\phi \cdot r \cdot \sin\theta$	$\frac{\partial z}{\partial \phi} = 0$

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial y}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\phi\cos\theta & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\sin\phi\cos\theta & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$\sin\theta\cos\phi\left[0 + r^2\sin^2\theta\cos\phi\right] - r\cos\phi\cos\theta\left[0 - r\sin\theta\cos\theta\cos\phi\right] - r\sin\theta\sin\phi\left[-r\sin^2\theta\sin\phi - r\sin\phi\cos^2\theta\right].$$

$$= r^2\sin^3\theta\cos^2\phi + r^2\cos^2\phi\cos^2\theta\sin\theta + r^2\sin^2\phi\sin\theta(\sin^2\theta + \cos^2\theta)$$

$$= r^2\sin\theta(\sin^2\theta\cos^2\phi + \cos^2\phi\cos^2\theta + \sin^2\phi)$$

$$= r^2\sin\theta(\cos^2\phi(\sin^2\theta + \cos^2\theta) + \sin^2\phi)$$

$$= r^2\sin\theta(\cos^2\phi + \sin^2\phi)$$

$$= r^2\sin\theta$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$$

06/10/2017

***5. If $x+y+z=u$ $y+z=v$ $z=uvw$ then find the value of $\frac{\partial(x,y,z)}{\partial(u,v,w)} =$

$x+y+z=u$ $y+z=v$ $z=uvw$.

$x = u-(y+z)$	$y = v-z$	$z = uvw$
$x = u-v$	$y = v-uvw$	$\frac{\partial z}{\partial u} = vw$
$\frac{\partial x}{\partial u} = 1$	$\frac{\partial y}{\partial u} = -vw$	$\frac{\partial z}{\partial y} = uw$
$\frac{\partial x}{\partial v} = -1$	$\frac{\partial y}{\partial v} = 1-uw$	
$\frac{\partial x}{\partial w} = 0$	$\frac{\partial y}{\partial w} = -uv$	$\frac{\partial z}{\partial w} = uv$

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 \\ -vw & 1-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$1((1-uw)uv + u^2vw) + 1$

$(-uv^2w + uv^2w] + 0$

$= uv - u^2wv + u^2vw - uv^2w$
$+ uv^2w$

$= uv$

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = uv$$

* Total differntiation.

$$du = \frac{\partial u}{\partial x} \times dx + \frac{\partial u}{\partial y} + \partial y.$$

If $u = f(x, y)$, $x = x(t)$ and $y = y(t)$. then the total derivative of u w.r.t t is $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$

The total derivative is also called as excert derivative.

Q.1 find the total or excert derivative of the following.

1. $u = x^3 + xy^2 + x^2y + y^3$
2. $r = x \sin\theta \cos\phi$
3. $z = xy^2 + x^2y$ where $x = at$ and $y = 2at$
4. $u = xy + yz + zx$, where $x = t\cos t$, $y = t\sin t$, $z = t$ at $\pi/4$.

1. $u = x^3 + xy^2 + x^2y + y^3$

Given $u = x^3 + xy^2 + x^2y + y^3$

Diffⁿ u partially w.r.t x and y

$$\frac{\partial y}{\partial x} = 3x^2+y^2+2xy$$

$$\frac{\partial u}{\partial y} = 2xy+x^2+3y^2$$

w.k.t

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$du = (3x^2+y^2+2xy)\,dx + (2xy+x^2+3y^2)\,dy$$ //

2. Given $x = r\sin\theta\cos\phi$

Diffn x partially w.r.t r, θ and ϕ

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi\times 1$$

$$\frac{\partial x}{\partial \theta} = r\cos\phi\cos\theta$$

$$\frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi$$

w.k.t

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta + \frac{\partial x}{\partial \phi}d\phi$$

$$dx = (\sin\theta\cos\phi)\,dr + (r\cos\phi\cos\theta)\,d\theta - (r\sin\theta\sin\phi)\,d\phi$$ //

3. Given $z = xy^2+x^2y$ → ① when $x = at$ and $y = 2at$

Diffn eqn ① partially w.r.t x and y

$$\frac{\partial z}{\partial x} = y^2+2xy = (2at)^2+2(at)(2at) = 8a^2t^2$$

$$\frac{\partial z}{\partial y} = 2xy+x^2 = 2(at)(2at)+(at)^2 = 5a^2t^2$$ //.

further we have

$x = at$ $\quad\quad$ $y = 2at$

$\frac{dx}{dt} = a$ // and $\frac{dy}{dt} = 2a$ //

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w.k.t

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (8a^2t^2)(a) + (5a^2t^2)(2a) = 18a^3t^2$$

(4) Given $u = xy + yz + zx$

$$\frac{\partial u}{\partial x} = y + z = (t\sin t + t)$$

$$\frac{\partial u}{\partial y} = x + z = (t\cos t + t)$$

$$\frac{\partial u}{\partial z} = y + x = (t\sin t + t\cos t)$$

$x = t\cos t$	$y = t\sin t$	$z = t$
$\frac{dx}{dt} = \cos t - t\sin t$	$\frac{dy}{dt} = \sin t + t\cos t$	$\frac{dz}{dt} = 1$

w.k.t

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

at $t = \pi/4$.

$$\frac{du}{dt} = \left(\frac{\pi}{4\sqrt{2}} + \pi/4\right)\left(1/\sqrt{2} - \frac{\pi}{4\sqrt{2}}\right) + \left(\frac{\pi}{4\sqrt{2}} + \frac{\pi}{4}\right)\left(\frac{1}{\sqrt{2}} + \pi/4\sqrt{2}\right) +$$

$$\left(\frac{\pi}{4\sqrt{2}} + \frac{\pi}{4\sqrt{2}}\right)(1)$$

$$\frac{du}{dt} = \left(\pi/4\sqrt{2} + \pi/4\right)\left(\sqrt{2}\right) + \frac{\pi}{2\sqrt{2}}$$

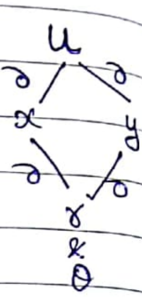
* <u>Differentiation of composite function.</u>

09/10/2017 * <u>chain rule of partial differentiation.</u>

If $u = f(r, y)$ where, $x = f(r, \theta)$ and $y = f(r, \theta)$. that is u is function of x and y and x and y are the functions of r, θ then the partial derivative of u is with respect to r and θ

are defined as $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r}$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \theta}$$



are called chain rule of partial differentiation.

Problems :-

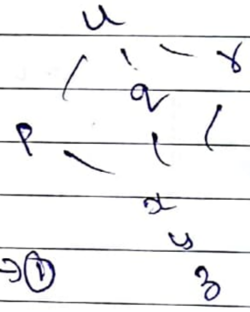
** *Q.1* ** If $u = f(x/y, y/z, z/x)$ then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial y} = 0$

$u = f(x/y, y/z, z/x)$

$p = x/y \quad q = y/z \quad r = z/x.$

$u = f(p, q, r)$

By chain rule of partial diffⁿ we get.



$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial x} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} \\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial y} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial y} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y}. \\ \frac{\partial u}{\partial z} &= \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial z} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial z} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial z}. \end{aligned}\right\} \rightarrow (1)$$

$p = x/y$	$q = y/z$	$r = z/x$
$\frac{\partial p}{\partial x} = 1/y$	$\frac{\partial q}{\partial x} = 0$	$\frac{\partial r}{\partial x} = -z/x^2$
$\frac{\partial p}{\partial y} = -x/y^2$	$\frac{\partial q}{\partial y} = 1/z$	$\frac{\partial r}{\partial y} = 0$
$\frac{\partial p}{\partial z} = 0$	$\frac{\partial q}{\partial z} = -y/z^2$	$\frac{\partial r}{\partial z} = \frac{1}{x}$

Sub in $\rightarrow (1)$

$\frac{\partial u}{\partial x} = \frac{1}{y}\frac{\partial u}{\partial p} - \frac{z}{x^2}\frac{\partial u}{\partial r} \Rightarrow x \cdot \frac{\partial u}{\partial x} = \frac{x}{y}\frac{\partial u}{\partial p} - z/x \frac{\partial u}{\partial r} \rightarrow (2)$

$\frac{\partial u}{\partial y} = -x/y^2 \frac{\partial u}{\partial p} + \frac{1}{z}\frac{\partial u}{\partial q} \Rightarrow y \cdot \frac{\partial u}{\partial y} = -x/y \frac{\partial u}{\partial p} + y/z \frac{\partial u}{\partial q} \rightarrow (3)$

$$\frac{\partial u}{\partial z} = -y/z^2 \frac{\partial u}{\partial q} + \frac{1}{z}\frac{\partial u}{\partial r} \Rightarrow z\frac{\partial u}{\partial z} = -y/z \frac{\partial u}{\partial q} + z/x \frac{\partial u}{\partial r} \rightarrow (4)$$

Adding (2) (3) and (4), we get

$$x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.$$

10/10/2017

Q2. If $z=(x,y)$, $x=u-v$ and $y=uv$ then P.T

(i) $(u+v)\frac{\partial z}{\partial x} = u\cdot\frac{\partial z}{\partial x} - v\frac{\partial z}{\partial v}$

(ii) $(u+v)\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$

By chain rule of P.D

$$\left.\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}\end{aligned}\right\} \rightarrow (1)$$

We have.

$x = u-v$	$y = uv$
$\frac{\partial x}{\partial u} = 1$	$\frac{\partial y}{\partial u} = v$
$\frac{\partial x}{\partial v} = -1$	$\frac{\partial y}{\partial v} = u$

Sub in (1).

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} + v\frac{\partial z}{\partial y} \rightarrow (2)$$

$$\frac{\partial z}{\partial v} = -\frac{\partial z}{\partial x} + u\frac{\partial z}{\partial y} \rightarrow (3).$$

(i) $(u\times eq^n(2)) - (v\times eq^n(3))$

$$u\cdot\frac{\partial z}{\partial u} - v\frac{\partial z}{\partial v} = u\frac{\partial z}{\partial x} + uv\frac{\partial z}{\partial y} + v\frac{\partial z}{\partial x} - uv\frac{\partial z}{\partial y}$$

$$= (u+v)\frac{\partial z}{\partial x}$$

ii) adding eq^n ② and ③

$$\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = v\frac{\partial z}{\partial y} + u\frac{\partial z}{\partial y}$$

$$= (v+u)\frac{\partial z}{\partial y} \quad //.$$

(3) If $z = f(x,y)$ where $x = u^2 - v^2$ and $y = 2uv$ then prove that

$$\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = 4(u^2+v^2)\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]$$

⇒ By chain rule of PD.

$$\left.\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}\end{aligned}\right\} \rightarrow ①$$

we have.

$x = u^2 - v^2$	$y = 2uv$
$\frac{\partial x}{\partial u} = 2u$	$\frac{\partial y}{\partial u} = 2v$
$\frac{\partial x}{\partial v} = -2v$	$\frac{\partial y}{\partial v} = 2u$

Sub in (1).

$$\frac{\partial z}{\partial u} = 2u\frac{\partial z}{\partial x} + 2v\frac{\partial z}{\partial y} = 2\left(u\cdot\frac{\partial z}{\partial x} + v\cdot\frac{\partial z}{\partial y}\right) \rightarrow ②$$

$$\frac{\partial z}{\partial v} = -2v\frac{\partial z}{\partial x} + 2u\frac{\partial z}{\partial y} = 2\left(u\frac{\partial z}{\partial x} - v\frac{\partial z}{\partial x}\right) \rightarrow ③$$

squaring and adding eq^n ② and ③

$$\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = 4\left[u\frac{dz}{\partial x} + v\frac{\partial z}{\partial y}\right]^2 + 4\left[u\frac{\partial z}{\partial y} - v\cdot\frac{\partial z}{\partial x}\right]^2$$

$$= 4\left\{u^2\left(\frac{\partial z}{\partial x}\right)^2 + v^2\left(\frac{\partial z}{\partial y}\right)^2 + 2uv\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + u^2\left(\frac{\partial z}{\partial y}\right)^2 + v^2\left(\frac{\partial z}{\partial x}\right)^2 - 2uv\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\right]$$

$$= 4\left\{u^2\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right] + v^2\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]\right\}$$

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$$= 4(u^2+v^2)\left\{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right\}$$

**

(4) If $z=f(x,y)$ where $x=e^u \sin v$ and $y=e^u \cos v$ then prove that.

$$\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2 = e^{2u}\left\{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right\}.$$

By chain rule of P.D

$$\left.\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}\\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}\end{aligned}\right\} \to (1)$$

we have.

$x=e^u \sin v$	$y=e^u \cos v$.
$\frac{\partial x}{\partial u} = e^u \sin v$	$\frac{\partial y}{\partial u} = e^u \cos v$
$\frac{\partial x}{\partial v} = e^u \cos v$	$\frac{\partial y}{\partial v} = -e^u \sin v$

Sub in (1).

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot e^u\sin v + \frac{\partial z}{\partial y}\cdot e^u\cos v = e^u\left[\left(\sin v\frac{\partial z}{\partial x}\right)+\left(\cos v\frac{\partial z}{\partial y}\right)\right] \to (2)$$

$$\frac{\partial z}{\partial v} = \frac{\partial t}{\partial x}\cdot e^u\cos v - \frac{\partial z}{\partial y}\cdot e^u\sin v = e^u\left[\left(\cos v\frac{\partial z}{\partial x}\right)-\left(\sin v\frac{\partial z}{\partial y}\right)\right] \to (3)$$

squaring and adding eq^n (2) and (3)

$$\left(\frac{\partial z}{\partial u}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2 = e^{2u}\left\{\left[\left(\sin v\frac{\partial z}{\partial x}\right)+\left(\cos v\frac{\partial z}{\partial y}\right)\right]^2 + \left[\cos v\frac{\partial z}{\partial x} - \sin v\frac{\partial z}{\partial y}\right]^2\right\}$$

$$= e^{2u}\left\{\left(\sin^2 v\left(\frac{\partial z}{\partial x}\right)^2 + \cos^2 v\left(\frac{\partial z}{\partial y}\right)^2 + 2\sin v\cos v\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\right) + \left(\cos^2 v\left(\frac{\partial z}{\partial x}\right)^2 + \sin^2 v\left(\frac{\partial z}{\partial y}\right)^2 - 2\sin v\cos v\frac{\partial z}{\partial y}\frac{\partial z}{\partial x}\right)\right\}.$$

$$= e^{2u}\left\{\left(\frac{\partial z}{\partial x}\right)^2\left(\sin^2 v+\cos^2 v\right) + \left(\frac{\partial z}{\partial y}\right)^2\left(\sin^2 v+\cos^2 v\right)\right\}$$

$$= e^{2u}\left\{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right\} //.$$

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* Homogenous functions and Euler's theorm

* Homogenous function

- A function $u=f(x,y)$ is said to be homogenous function. if it can be expressed as $u=x^n f(y/x)$ (or) $u=y^n g(x/y)$
- where n is a degree of homogenous function

ex:-

$$u=\frac{x^3+y^3}{\sqrt{x+y}} = \frac{x^3\left(1+\frac{y^3}{x^3}\right)}{\sqrt{x}\left(\sqrt{1+y/x}\right)}$$

$$= x^{3-1/2}\left\{\frac{1+(y/x)^3}{\sqrt{1+(y/x)}}\right\}$$

$$= u = x^{5/2}\left\{\frac{1+(y/x)^3}{\sqrt{1+(y/x)}}\right\}$$

= u is a HF of degree $n=5/2$.

~~By Euler's theorem $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$ = 5/2 u~~

* Euler's theorm.

If $u=f(x,y)$ is a homogenous function of degree n then $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$

* Euler's extension theorem:

$$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y}=n(n-1)u.$$

(or)

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^2 u=n(n-1)u.$$

1: If $u=\frac{x^3+y^3}{\sqrt{x+y}}$ then prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=5/2\,u$

$$u=\frac{x^3+y^3}{\sqrt{x+y}} = \frac{x^3\left(1+y^3/x^3\right)}{\sqrt{x}\left(\sqrt{1+y/x}\right)}$$

$$x^{3-1/2}\left\{\frac{1+(y/x)^3}{\sqrt{1+(y/x)}}\right\}$$

$$u = x^{5/2}\left\{\frac{1+(y/x)^3}{\sqrt{1+(y/x)}}\right\}$$

u is a H.F of degree $n = 5/2$.

By Euler's theorem.

$$x\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = nu$$

$$= 5/2\, u.$$

(2) If $u = \left(\frac{z^4}{x^3+y^3}\right)^{1/3}$ then prove that $3(xu_x + yu_y + zu_z) = u$.

$$u = \left(\frac{z^4}{x^3+y^3}\right)^{1/3}$$

$$= \left\{\frac{x^4(z^4/x^4)}{x^3(1+y^3/x^3)}\right\}^{1/3}$$

$$= \left\{\frac{x(z/x)^4}{1+(y/x)^3}\right\}^{1/3}.$$

$$u = x^{1/3}\left\{\frac{(z/x)^4}{1+(y/x)^3}\right\}^{1/3}$$

u is a H.F of degree $n = 1/3$.

By Euler's theorem.

$$x\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu = 1/3\, u$$

$$= 3(xu_x + yu_y + zu_z) = u.//$$

(3) If $u = \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{(x+y)}$ then prove that $xu_x + yu_y + zu_z = 0$

$$u = \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}.$$

$$= \frac{x}{x(y/x+z/x)} + \frac{x(y/x)}{x(z/x+1)} + \frac{x(z/x)}{x(1+y/x)}$$

$$u = x^0 \times \left\{\frac{1}{(y/x+z/x)} + \frac{y/x}{(1+z/x)} + \frac{z/x}{(1+y/x)}\right\}$$

u is a HF of degree $n = 0$

By Euler's theorem

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu$$

$$=0\times 4.$$

$$x\frac{\partial u}{\partial x}+y\cdot\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0.$$

(4) If $u=\log\left(\frac{x^4+y^4}{x+y}\right)$ then prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3.$

Given $u=\log\left(\frac{x^4+y^4}{x+y}\right)$

$$e^u=\frac{x^4+y^4}{x+y}.$$

let $e^u=z$

$$\therefore\ z=\frac{x^4+y^4}{x+y}$$

$$z=\frac{x^4\left(1+y^4/x^4\right)}{x\left(1+y/x\right)}\Rightarrow x^3\left\{\frac{1+(y/x)^4}{1+(y/x)}\right\}$$

$\therefore z$ is a H.F of degree $n=3$

By Euler's theorem.

$$x\cdot\frac{\partial z}{\partial x}+y\cdot\frac{\partial z}{\partial y}=nz$$

put $z=e^u$ and $n=3.$

$$x\cdot\frac{\partial}{\partial x}(e^u)+y\frac{\partial}{\partial y}(e^u)=3\times e^u$$

$$xe^u\frac{\partial u}{\partial x}+ye^u\frac{\partial u}{\partial y}=3e^u$$

Dividing by e^u

$$x\cdot\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3.$$

(5) If $u=e^{\left(\frac{x^3y^3}{x^2+y^2}\right)}$ then P.T $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=4u\log u.$

$$u=e^{\frac{x^3y^3}{x^2+y^2}}$$

$$\log u=\frac{x^3y^3}{x^2+y^2}$$

let $\log u = z$

$$z = \frac{x^3 y^3}{x^2+y^2}$$

$$z = \frac{x^6 (y^3/x^3)}{x^2(1+y^2/x^2)} = x^4 \left\{ \frac{(y/x)^3}{1+(y/x)^2} \right\}$$

z is a HF of degree $n=4$.

By Euler's theorem.

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$

put $z = \log u$ and $n=4$.

$$x \cdot \frac{\partial}{\partial x}(\log u) + y \frac{\partial}{\partial y}(\log u) = 4 \log u.$$

$$x \times \frac{1}{u} \times \frac{\partial u}{\partial x} + y \times \frac{1}{u} \frac{\partial u}{\partial y} = 4 \log u$$

ⓧ^{ly} by u

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = 4u \log u$$ //.

***(6) If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$ then prove that (i) $x u_x + y u_y = \sin 2u$.

(ii) $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = \sin 4u - \sin 2u$.

Given $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

$$\tan u = \frac{x^3+y^3}{x-y}.$$

let $\tan u = z$

$$z = \frac{x^3+y^3}{x-y} = \frac{x^3(1+y^3/x^3)}{x(1-y/x)}$$

$$= x^2 \left\{ \frac{1+(y/x)^3}{1-(y/x)} \right\}$$

z is a HF of degree $n=2$

(i) By Euler's theorem, we get

$$x \cdot \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = nz$$

put $z = \tan u$ and $n = 2$

$$x \cdot \frac{\partial (\tan u)}{\partial x} + y \frac{\partial (\tan u)}{\partial y} = 2 \tan u.$$

$$x \sec^2 u \cdot \frac{\partial u}{\partial x} + y \cdot \sec^2 u \cdot \frac{\partial u}{\partial y} = 2 \tan u$$

Dividing by $\sec^2 u$

$$x \cdot \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{2 \tan u}{\sec^2 u}$$

$$= \frac{2 \sin u}{\cos u} \times \cos^2 u$$

$$= 2 \sin u \cos u$$

$$x \cdot \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u \rightarrow (1)$$

(ii) Diff^n eq^n (1) partially w.r.t x

$$x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y \frac{\partial^2 u}{\partial x \partial y} = \cos 2u \times 2 \frac{\partial u}{\partial x}$$

(×)^{ly} by x.

$$x^2 \frac{\partial^2 u}{\partial x^2} + x \frac{\partial u}{\partial x} + xy \frac{\partial^2 u}{\partial x \partial y} = 2 \cos 2u \cdot x \frac{\partial u}{\partial x} \rightarrow (3)$$

III^{ly}

$$y^2 \frac{\partial^2 u}{\partial y^2} + y \cdot \frac{\partial u}{\partial y} + xy \frac{\partial^2 u}{\partial x \partial y} = 2 \cos 2u \cdot y \frac{\partial u}{\partial y} \rightarrow (4)$$

adding (3) (+) (4)

$$x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}\right) = 2 \cos 2u \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}\right)$$

using eq (1)

$$x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + \sin 2x = 2 \cos 2u \cdot \sin 2u$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} = \sin 4u - \sin 2u$$

(7) If $u = \operatorname{cosec}^{-1}\left(\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)$ then prove that $\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 u = \frac{\tan u}{6}\left(1+\frac{\sec^2 u}{6}\right)$

$$u = \operatorname{cosec}^{-1}\left(\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)$$

$$\operatorname{cosec} u = \frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}.$$

let $\operatorname{cosec} u = z$

$$z = \frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}} = \frac{x^{1/2}(1+y^{1/2}/x^{1/2})}{x^{1/3}(1+y^{1/3}/x^{1/3})}$$

$$z = x^{1/6}\left\{\frac{1+(y/x)^{1/2}}{1+(y/x)^{1/3}}\right\}$$

z is a HF of degree $n = 1/6$

By Euler's theorem, we get.

$$x\cdot\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$

put $z = \operatorname{cosec} u$ and $n = 1/6$

$$x\frac{\partial}{\partial x}(\operatorname{cosec} u) + y\frac{\partial}{\partial y}(\operatorname{cosec} u) = 1/6 \operatorname{cosec} u$$

$$-x \operatorname{cosec} u\cdot\cot u\cdot\frac{\partial u}{\partial x} - y\operatorname{cosec} u\cdot\cot u\frac{\partial u}{\partial y} = 1/6 \operatorname{cosec} u.$$

Dividing by $(-\operatorname{cosec} u\cdot\cot u)$

$$x\cdot\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1/6\operatorname{cosec} u}{-\operatorname{cosec} u\cdot\cot u}$$

$$x\cdot\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -1/6\tan u \rightarrow (1)$$

Diffn eqn (1) partially w.r.t x.

$$x\cdot\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\partial y} = -1/6 \times \sec^2 u \times \frac{\partial u}{\partial x}.$$

(X)ly by x

$$x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + xy\frac{\partial^2 u}{\partial x\partial y} = -1/6\sec^2 u \times x\frac{\partial u}{\partial x} \rightarrow (3)$$

$\frac{1}{1}^{ly}$.

$$y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} + xy\frac{\partial^2 u}{\partial x\partial y} = -\frac{1}{6}\sec^2 u \cdot y\frac{\partial u}{\partial y} \rightarrow (4)$$

adding (3) and (4)

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + \left(x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y}\right) = -\frac{1}{6}\sec^2 u \times \left[x\frac{\partial u}{\partial x} + y\right.$$

using eqn (1)

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} - \frac{1}{6}\tan u = -\frac{1}{6}\sec^2 u\left(-\frac{1}{6}\tan u\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} = \frac{\sec^2 u}{6}\left(\frac{\tan u}{6}\right) + \frac{\tan u}{6}$$

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 u = \frac{\tan u}{6}\left(1 + \frac{\sec^2 u}{6}\right)$$

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Department of Mathematics

Lecture Notes
Calculus & Linear Algebra (18MAT11)

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CALCULUS AND LINEAR ALGEBRA

MODULE - 03

INTEGRAL CALCULUS

Note :-

$$\int x^n \, dx = \frac{x^{n+1}}{n+1}$$

$$\int \sin x \, dx = -\cos x$$

$$\int \cos x \, dx = \sin x$$

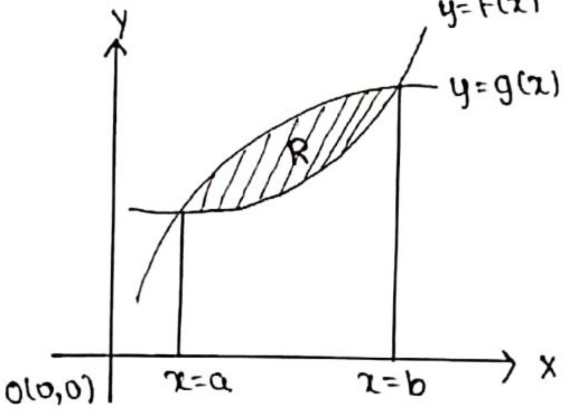
$$\int e^{ax} = \frac{e^{ax}}{a}$$

$$\int u.v \, dx = u\int v \, dx - \int (u' \int v \, dx) \, dx$$

Multiple Integrals:

1. Integral $\int_{x=a}^{b} f(x) \, dx$ can be described as the length of a curve $y = f(x)$ from $x=a$ to $x=b$ also it is called the line integral.

2. Integral $\int_a^b \int_{y=f(x)}^{g(x)} \phi(x,y)\, dy\, dx$ can be described as region of Surface bounded between $y=f(x)$, $y=g(x)$ and $x=a$, $x=b$



3. In Integral,

$$I = \int_{x=a}^{b} \int_{y=f(x)}^{g(x)} \int_{z=h_1(x,y)}^{h_2(x,y)} \phi(x,y,z)\, dz\, dy\, dx$$ used to calculate the volume between the mentioned boundaries

1. Evaluate $\int_0^1 \int_0^x (x^2+y^2)\, dy\, dx$

Let $I = \int_0^1 \int_0^x (x^2+y^2)\, dy\, dx$

$$= \int_{x=0}^{1} \left[\int_{y=0}^{x} (x^2+y^2)\, dy \right] dx$$

$$= \int_{x=0}^{1} \left[x^2 y + \frac{y^3}{3} \right]_{y=0}^{x} dx$$

$$= \int_{x=0}^{1} \left[x^3 + \frac{x^3}{3} \right] dx$$

$$= \int_0^1 \frac{4x^3}{3}\, dx$$

$$= \frac{4}{3}\int_0^1 x^3\, dx$$

$$= \frac{4}{3}\left[\frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{3}\left[x^4\right]_0^1$$

$$= \frac{1}{3}[1-0]$$

$$I = \frac{1}{3}$$

2. Evaluate $\int_0^1 \int_x^{\sqrt{x}} xy\, dy\, dx$

Let $I = \int_0^1 \int_x^{\sqrt{x}} xy\, dy\, dx$

$$= \int_{x=0}^1 \int_{y=x}^{\sqrt{x}} xy\, dy\, dx$$

$$= \int_{x=0}^1 x \int_{y=x}^{\sqrt{x}} y\, dy\, dx$$

$$= \int_{x=0}^1 x \int_{y=x}^{\sqrt{x}} y\, dy\, dx$$

$$= \int_{x=0}^1 x \left[\frac{y^2}{2}\right]_x^{\sqrt{x}} dx$$

$$= \frac{1}{2}\int_{x=0}^1 x \left[y^2\right]_x^{\sqrt{x}} dx$$

$$= \frac{1}{2}\int_{x=0}^{1} x(x-x^2)\,dx$$

$$= \frac{1}{2}\int_{x=0}^{1} (x^2-x^3)\,dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{3}-\frac{1}{4}\right]$$

$$= \frac{1}{2}\left[\frac{4-3}{12}\right]$$

$$I = \frac{1}{24}$$

3. Evaluate $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(x^2+y^2+z^2)\,dz\,dy\,dx$

$$I = \int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(x^2+y^2+z^2)\,dz\,dy\,dx$$

$$= \int_{x=-c}^{c}\int_{y=-b}^{b}\left[x^2z+y^2z+\frac{z^3}{3}\right]_{z=-a}^{a}dy\,dx$$

$$= \int_{x=-c}^{c}\int_{y=-b}^{b}\left\{\left[ax^2+ay^2+\frac{a^3}{3}\right]-\left[-ax^2-ay^2-\frac{a^3}{3}\right]\right\}dy\,dx$$

$$= \int_{x=-c}^{c}\int_{y=-b}^{b}\left[2ax^2+2ay^2+\frac{2a^3}{3}\right]dy\,dx$$

$$= 2a\int_{x=-c}^{c}\int_{y=-b}^{b}\left[x^2+y^2+\frac{a^2}{3}\right]dy\,dx$$

$$= 2a\int_{x=-c}^{c}\left[x^2y+\frac{y^3}{3}+\frac{a^2y}{3}\right]_{y=-b}^{b}dx$$

$$= 2a\int_{x=-c}^{c}\left\{\left[bx^2+\frac{b^3}{3}+\frac{ba^2}{3}\right]-\left[-\frac{bx^2}{1}-\frac{b^3}{3}-\frac{ba^2}{3}\right]\right\}dx$$

$$= 2a\int_{x=-c}^{c}\left[2bx^2+\frac{2b^3}{3}+\frac{2ba^2}{3}\right]dx$$

$$= 4ab\int_{x=-c}^{c}\left[x^2+\frac{b^2}{3}+\frac{a^2}{3}\right]dx$$

$$= 4ab\left[\frac{x^3}{3}+\frac{b^2}{3}x+\frac{a^2x}{3}\right]_{-c}^{c}$$

$$= 4ab\left\{\left[\frac{c^3}{3}+\frac{cb^2}{3}+\frac{ca^2}{3}\right]-\left[-\frac{c^3}{3}-\frac{cb^2}{3}-\frac{ca^2}{3}\right]\right\}$$

$$= 4ab\left[\frac{2c^3}{3}+\frac{2cb^2}{3}+\frac{2ca^2}{3}\right]$$

$$= 8abc\left[\frac{a^2}{3}+\frac{b^2}{3}+\frac{c^2}{3}\right]$$

$$= \frac{8}{3}abc\left[a^2+b^2+c^2\right]$$

4. Evaluate $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dy\,dx\,dz$

$$I = \int_{z=-1}^{1}\int_{x=0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dy\,dx\,dz$$

$$I = \int_{z=-1}^{1}\int_{x=0}^{z}\left[xy+\frac{y^2}{2}+yz\right]_{y=x-z}^{x+z}dx\,dz$$

$$= \int_{z=-1}^{1}\int_{x=0}^{z}\left\{x(x+z)+\frac{(x+z)^2}{2}+z(x+z)\right]-\left[x(x-z)+\frac{(x-z)^2}{2}+z(x-z)\right]\right\}dx\,dz$$

$$= \int_{z=-1}^{1}\int_{x=0}^{z}\left\{x(x+z-x+z)+\frac{1}{2}\left[(x+z)^2-(x-z)^2+z(x+z)-x+z)\right\}dx\,dz$$

$$= \int_{z=-1}^{1} \int_{x=0}^{2} (4zx + 2z^2)\, dx\, dz$$

$$= \int_{z=-1}^{1} \left[4z \frac{x^2}{2} + 2z^2 x\right]_{x=0}^{2} dz$$

$$= \int_{z=-1}^{1} (2z^3 + 2z^3) dz$$

$$= \int_{z=-1}^{1} 4z^3 dz$$

$$= 4\left[\frac{z^4}{4}\right]_{-1}^{1}$$

$$= \left[z^4\right]_{-1}^{1}$$

$$= (1^4) - (-1)^4$$

$$= 1 - 1$$

$$\boxed{I = 0}$$

5. Evaluate $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$

$$I = \int_{x=0}^{a} \int_{y=0}^{x} e^{x+y}\, e^z\, dz\, dy\, dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{x} e^{x+y} \int_{z=0}^{x+y} e^z\, dz\, dy\, dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{x} e^{x+y} \left[e^z\right]_{z=0}^{x+y} dy\, dx$$

$$= \int_{x=0}^{a} \int_{0}^{x} e^{x+y} \left[e^{x+y} - 1\right] dy\, dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{x} \left[e^{2x+2y} - e^{x+y}\right] dy\, dx$$

$$= \int_{x=0}^{a} e^{2x} \int_{y=0}^{x} e^{2y}\, dy\, dx - \int_{x=0}^{a} e^{x} \int_{y=0}^{x} e^{y}\, dy \cdot dx$$

$$= \int_{x=0}^{a} e^{2x} \int_{y=0}^{x} e^{2y}\, dy\, dx - \int_{x=0}^{a} e^{x} \int_{y=0}^{x} e^{y}\, dy \cdot dx$$

$$= \frac{1}{2} \int_{x=0}^{a} e^{2x} \left[2^{2y} \int_{0}^{x} dx - \int_{x=0}^{a} e^{x} \left[e^{y} \right]_{0}^{x} dx$$

$$= \frac{1}{2} \int_{0}^{a} (e^{4x} - e^{2x})\, dx - \int_{0}^{a} (e^{2x} - e^{x})\, dx$$

$$= \frac{1}{2} \left[\frac{e^{4x}}{4} - \frac{e^{2x}}{2} \right]_{0}^{a} - \left[\frac{e^{2x}}{2} - \frac{e^{x}}{1} \right]_{0}^{a}$$

$$= \frac{1}{2} \left\{ \left[\frac{e^{4a}}{4} - \frac{e^{2a}}{2} \right] - \left[\frac{1}{4} - \frac{1}{2} \right] \right\} - \left\{ \left[\frac{e^{2a}}{2} - \frac{e^{a}}{1} \right] - \left[\frac{1}{2} - 1 \right] \right\}$$

$$= \frac{1}{2} \left\{ \frac{e^{4a}}{4} - \frac{e^{2a}}{2} + \frac{1}{4} \right\} - \left\{ \frac{e^{2a}}{2} - e^{a} + \frac{1}{2} \right\}$$

$$= \frac{e^{4a}}{8} + \frac{e^{2a}}{4} + \frac{1}{8} - \frac{e^{2a}}{2} + e^{a} - \frac{1}{2}$$

$$= \frac{e^{4a}}{8} - \frac{3}{4} e^{2a} + e^{a} \cdot \frac{3}{8}$$

$$I = \frac{1}{8} \left[e^{4a} - 6a^{2a} + 8e^{a} - 3 \right]$$

6. Evaluate $\int_{0}^{a} \int_{0}^{\sqrt{a^2-x^2}} \int_{0}^{\sqrt{a^2-x^2-y^2}} \frac{1}{\sqrt{a^2-x^2-y^2-z^2}}\, dz\, dy\, dx$

$$I = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} \frac{1}{\sqrt{a^2-x^2-y^2-z^2}}\, dz\, dy\, dx$$

$$I = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} \frac{1}{\left(\sqrt{a^2-x^2-y^2}\right)^2 - z^2}\, dz\, dy\, dx$$

$$I = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} \frac{1}{\left(\sqrt{(a^2-x^2-y^2)}\right)^2 - z^2} \, dz \, dy \, dx$$

Let $K = \sqrt{a^2-x^2-y^2}$

$$I = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{K} \frac{1}{\sqrt{K^2-z^2}} \, dz \, dy \, dz$$

$$= \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \left[\sin^{-1}(z/K) \right]_{z=0}^{K} dy \, dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \left[\sin^{-1}(1) - \sin^{-1}(0) \right] dy \, dx$$

$$= \frac{\pi}{2} \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} 1 \, dy \, dx$$

$$= \frac{\pi}{2} \int_{x=0}^{a} \left[y \right]_0^{\sqrt{a^2-x^2}} dx$$

$$I = \frac{\pi}{2} \int_0^a \sqrt{a^2-x^2} \, dx$$

Let $x = a \sin\theta \Rightarrow \theta = \sin^{-1}(x/a)$

$dx = d \cos\theta \, d\theta$

U.L $\Rightarrow x = a \Rightarrow \theta = \frac{\pi}{2}$

L.L $\Rightarrow x = 0 \Rightarrow \theta = 0$

$$\therefore I = \frac{\pi}{2} \int_0^{\pi/2} \sqrt{a^2 - a^2 \sin^2\theta} \, . \, a\cos\theta \, d\theta$$

$$= \frac{\pi}{2} \int_0^{\pi/2} a\cos\theta \, . \, a\cos\theta \, d\theta$$

$$= \frac{\pi a^2}{2} \int_0^{\pi/2} \cos^2\theta \, d\theta$$

$$= \frac{\pi a^2}{2} \int_0^{\pi/2} \left(\frac{1+\cos\theta}{2} \right) d\theta$$

$$= \frac{\pi a^2}{4} \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/2}$$

$$= \frac{\pi a^2}{4} \left(\frac{\pi}{2} + 0 \right) - (0+0)$$

$$I = \frac{\pi^2 a^2}{8}$$

7. Evaluate $\int_0^1 \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{1}{\sqrt{a^2-x^2-y^2-z^2}} \, dz \, dy \, dx$

$$I = \int_0^1 \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{1}{\sqrt{a^2-x^2-y^2-z^2}} \, dz \, dy \, dx$$

$$= \int_{x=0}^1 \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} \frac{1}{\left(\sqrt{a^2-x^2-y^2}\right)^2 - z^2} \, dz \, dy \, dx$$

let $K = \sqrt{a^2-x^2-y^2}$

$$= \int_{x=0}^1 \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{K} \frac{1}{\sqrt{K^2-z^2}} \, dz \, dy \, dx$$

$$= \int_{x=0}^1 \int_{y=0}^{\sqrt{a^2-x^2}} \left[\sin^{-1}(z/K) \right]_{z=0}^{K} dy \, dx$$

$$= \int_{x=0}^1 \int_{y=0}^{\sqrt{a^2-x^2}} \left[\sin^{-1}(1) - \sin^{-1}(0) \right] dy \, (dx)$$

$$= \frac{\pi}{2} \int_{x=0}^{1} \int_{y=0}^{\sqrt{a^2-x^2}} 1 \; dy \, dx$$

$$= \frac{\pi}{2} \int_{x=0}^{1} [y]_0^{\sqrt{a^2-x^2}} \, dx$$

$$= \frac{\pi}{2} \int_{0}^{1} \sqrt{a^2-x^2} \, dx$$

Let $x = a \sin\theta \Rightarrow \theta = \sin^{-1}(x/a)$

$dx = a\cos\theta \, d\theta$

UL : $x = 1 \Rightarrow \theta = \frac{\pi}{2}$

LL : $x = 0 \Rightarrow \theta = 0$

$$= \frac{\pi}{2} \int_{0}^{\pi/2} \sqrt{1-\sin^2\theta} \cdot \cos\theta \, d\theta$$

$$= \frac{\pi}{2} \int_{0}^{\pi/2} \cos\theta \cos\theta \, d\theta$$

$$= \frac{\pi}{2} \int_{0}^{\pi/2} \left(\frac{1-\cos\theta}{2} \right) d\theta$$

$$= \frac{\pi}{2} \left[0 + \frac{\sin 2\theta}{2} \right]_0^{\pi/2}$$

$$= \frac{\pi}{4} \left[\frac{\pi}{2} + 0 \right] - (0+0)$$

$$I = \frac{\pi^2}{8}$$

8. Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} x\,y\,z \; dz.dy\,dx$

$$I = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} x\,y\,z \; dz \, dy \, dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} xy \left[\frac{z^2}{2} \right]_0^{\sqrt{a^2-x^2-y^2}} dx\, dy$$

$$= \frac{1}{2} \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} xy\, (a^2 - x^2 - y^2)\, dy\, dx$$

$$= \frac{1}{2} \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} (a^2 xy - x^3 y)\, dy\, dx$$

$$= \frac{1}{2} \int_{x=0}^{a} \left[a^2 x \cdot \frac{y^2}{2} - x^3 \frac{y^2}{2} - \frac{xy^4}{4} \right]_{y=0}^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2} \int_{x=0}^{a} \left[\frac{a^2 x}{2} (a^2 - x^2) - \frac{x^3}{2} (a^2 - x^2) - \frac{x}{4} (a^2 - x^2) \right] dx$$

$$= \frac{1}{2} \int_{x=0}^{a} (a^2 - x^2) \left(\frac{a^2 x}{2} - \frac{x^3}{2} \cdot \frac{x (a^2 - x^2)}{4} \right) dx$$

$$= \frac{1}{2} \int_{x=0}^{a} \frac{(a^2 - x^2)}{4} \left[2a^2 x - 2x^3 - a^2 x + x^3 \right] dx$$

$$= \frac{1}{8} \int_0^a (a^2 - x^2)(a^2 x - x^3)\, dx$$

$$= \frac{1}{2} \int_0^a (a^4 x - a^2 a^3 - a^2 x^3 + x^5)\, dx$$

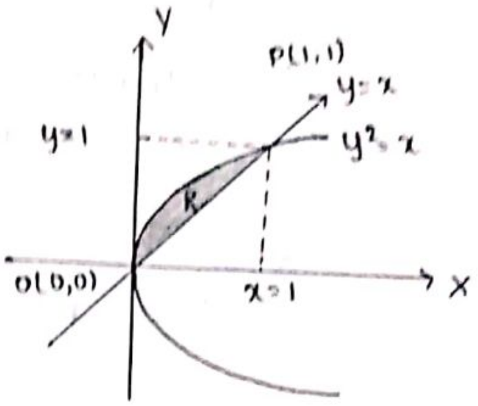
$$= \frac{1}{8} \int_0^a \left[x^5 - 2a^2 x^3 + a^4 x \right] dx$$

$$= \frac{1}{8} \left[\frac{x^6}{6} - \frac{2a^2 x^4}{4} + a^4 \frac{x^2}{2} \right]_0^a$$

$$= \frac{1}{8} \left\{ \frac{a^6}{6} - \frac{a^6}{2} + \frac{a^6}{2} \right\} \Rightarrow I = \frac{a^6}{48}$$

⇒ Change of Order of Integeration :-

1. Evaluate $\int_0^1 \int_x^{\sqrt{x}} xy \, dy \, dx$ by change of Order of Integeration.



$\Rightarrow I = \int_0^1 \int_x^{\sqrt{x}} xy \, dy \, dx$

here x varies as $x=0, x=1 \rightarrow (1)$

and then y varies as $y=x$

and $y=\sqrt{x} \Rightarrow y^2 = x \rightarrow (2)$

from (1) & (2)

$x = \sqrt{x}$

$\Rightarrow x^2 = x$

$\Rightarrow x^2 - x = 0$

$\Rightarrow x(x-1) = 0$

$\Rightarrow x = 0, x = 1$

$\Rightarrow y = 0, \ y = 1$

∴ By the change of order of Integral

$$I = \int_{y=0}^{1} \int_{x=y^2}^{y} xy \, dx \, dy$$

$$= \int_{y=0}^{1} y \int_{x=y^2}^{y} x \, dx \, dy$$

$$= \int_{y=0}^{1} y \left(\frac{x^2}{2} \right)_{y^2}^{y} dy$$

$$= \int_{y=0}^{1} y \left(\frac{y^2}{2} - \frac{y^4}{2} \right) dy$$

$$= \frac{1}{2} \int_0^1 (y^3 - y^5)\, dy$$

$$= \frac{1}{2} \left[\frac{y^4}{4} - \frac{y^6}{6} \right]_0^1$$

$$= \frac{1}{2} \left[\frac{1}{4} - \frac{1}{6} \right]$$

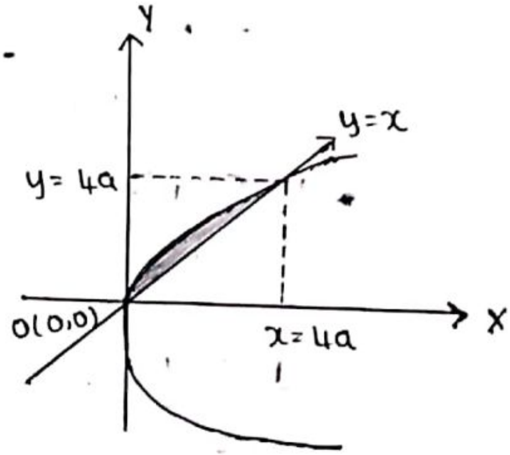
$$= \frac{1}{2} \left[\frac{6-4}{24} \right]$$

$$= \frac{1}{2} \times \frac{2}{24}$$

$$I = \frac{1}{24} \text{ Squnits.}$$

2. Evaluate by change of order of Integtration.

$$\int_0^{4a} \int_x^{2\sqrt{ax}} x^2\, dy\, dx \quad . \quad a>0 .$$



$$I = \int_{x=0}^{4a} \int_{y=x}^{2\sqrt{ax}} x^2\, dy\, dx \rightarrow (1)$$

Here , $x=0$, $x = 4a$

$y = x$, $y = 2\sqrt{ax}$, $\Rightarrow y^2 = 4ax$

$\rightarrow (1)$ $\qquad \rightarrow (2)$

From (1) and (2)

$$x = 2\sqrt{ax}$$

$$\Rightarrow x^2 = 4ax$$

$$\Rightarrow x^2 = 4ax = 0$$

$$\Rightarrow x(x - 4a) = 0$$

$$\Rightarrow x = 0 \ , \ x = 4a$$

$\Rightarrow y = 0 \ , \ y = 4a$

$$I = \int_{y=0}^{4a} \int_{x=y}^{y^2/4a} x^2 \, dx \, dy$$

$$= \int_{y=0}^{4a} \left[\frac{x^3}{3} \right]_{y}^{y^2/4a} dy$$

$$= \frac{1}{3} \int_{y=0}^{4a} \left[\left(\frac{y^2}{4a} \right)^3 - y^3 \right] dy$$

$$= \frac{1}{3} \int_{y=0}^{4a} \left[\frac{y^6}{64a^3} - y^3 \right] dy$$

$$= \frac{1}{3} \left[\frac{y^7}{7 \times 64a^3} - \frac{y^4}{4} \right]_0^{4a} dy$$

$$= \frac{1}{3} \left[\frac{(4a)^7}{7 \times 64a^3} - \frac{(4a)^4}{4} \right]$$

$$= \frac{1}{3} \left[\frac{4^7 \times a^7}{4^3 \times 7 \times a^3} - \frac{(4)^4 (a^4)}{4} \right]$$

$$= \frac{1}{3} \left[\frac{(4^5) a^4 - 7(4^4)(a^4)}{28} \right]$$

$$= \frac{1}{3} \left[\frac{256 a^4 - 1792 . a^4}{28} \right]$$

$$= \frac{1}{3} . a^4 \frac{[256 - 1792]}{28}$$

$$= \frac{a^4[-1536]}{28 \times 3}$$

$$= \frac{a^4[-1536]}{84}$$

$= a^4 \div 18.28$

$= \frac{1}{3}\left[\frac{(4^5)a^4 - 7(4^4)a^4}{28}\right]$

$= \frac{1}{3}[4^4]\left[\frac{4a^4 - 7a^4}{28}\right]$

$= \frac{1}{3}[4^4]\left[\frac{3a^4}{28}\right]$

$= \frac{16 \times 16 \times a^4}{28}$

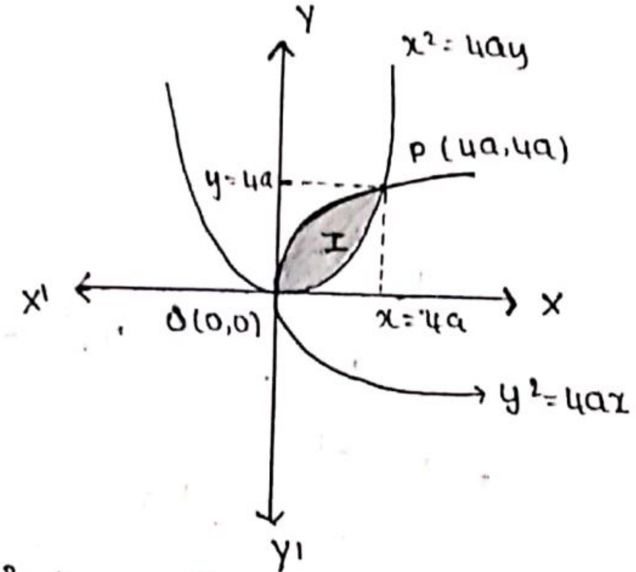
$\therefore$

$I = \frac{64a^4}{7}$ sq units

3. Evaluate $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} xy \, dy \, dx$ by changing the order of integration.

let $I = \int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} xy \, dy \, dx$

$I = \int_{x=0}^{4a}\int_{y=x^2/4a}^{2\sqrt{ax}} xy \, dy \, dx$



$x = 0$, $x = 4a$ and $y = \frac{x^2}{4a}$

$\Rightarrow x^2 = 4ay$, $y = 2\sqrt{ax} \Rightarrow y^2 = 4a \rightarrow$ (2)

From (1) and (2)

$\frac{x^2}{4a} = 2\sqrt{ax}$

$\Rightarrow x^2 = 8a\sqrt{ax}$

$x^4 = 64a^2(ax)$

$\Rightarrow x^4 - 64a^3x = 0$

$\Rightarrow x(x^3 - 64a^3) = 0$

$\Rightarrow x = 0 \;,\; x^3 = 64a^3$

$\Rightarrow x^3 = (4a)^3$

$\Rightarrow x = 4a$

where $x = 0 \Rightarrow y = 0$

$x = 4a \Rightarrow y = 4a$

$\therefore$ The change of order of integration we have

$$I = \int_{y=0}^{4a} \int_{x=y^2/4a}^{2\sqrt{ay}} xy \, dx\,dy$$

$$= \int_{y=0}^{4a} y \int_{y^2/4a}^{2\sqrt{ay}} x \, dx\,dy$$

$$= \int_{y=0}^{4a} y \left[\frac{x^2}{2} \right]_{y^2/4a}^{2\sqrt{ay}} dy$$

$$= \frac{1}{2} \int_{y=0}^{4a} \left(y \left[2\sqrt{ay} \right]^2 - \left(\frac{y^2}{4a} \right)^2 \right) dy$$

$$= \frac{1}{2} \int_0^{4a} y \left[4ay - \frac{y^4}{16a^2} \right] \Big) dy$$

$$= \frac{1}{2} \left\{ \frac{4ay^3}{3} - \frac{y^6}{6x/6a^2} \right\}_0^{4a}$$

$$= \frac{1}{2} \left\{ \frac{4a}{3} y^3 - \frac{y^6}{96a^2} \right\}_0^{4a}$$

$$= \frac{1}{2} \left[\frac{4a}{3} (4a)^3 - \frac{(4a)^6}{96a^2} \right]$$

$$= \frac{(4a)^4}{2}\left[\frac{1}{3} - \frac{16a^3}{96a^2}\right]$$

$$= \frac{(4a)^4}{2}\left[\frac{1}{3} - \frac{1}{6}\right]$$

$$= \frac{(4a)^4}{2} \times \frac{1}{6}$$

$$= \frac{(4a)^4}{12}$$

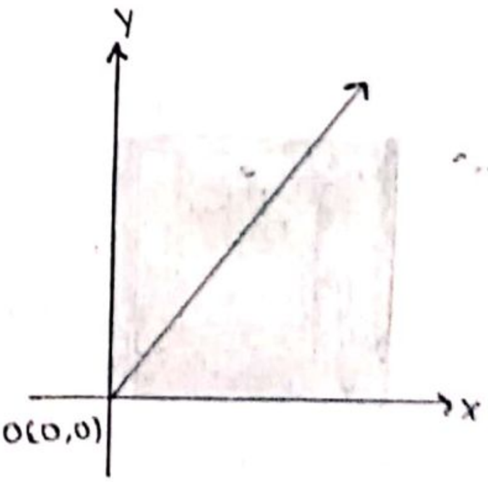
$$= \frac{4^4 \times a^4}{12}$$

$$= \frac{4 \times 4 \times 4 \times 4 \times a^4}{12}$$

$$I = \frac{64a^4}{3}$$

4. Evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$

$$I = \int_{x=0}^{\infty} \int_{y=x}^{\infty} \frac{e^{-y}}{y}\, dy\, dx$$



Here x varies from 0 to ∞ and y varies from x to ∞

$\therefore$ By the change of order of integration.

$$I = \int_{x=0}^{\infty} \int_{y=x}^{\infty} \frac{e^{-y}}{y}\, dy\, dx$$

$$I = \int_{y=0}^{\infty} \int_{x=0}^{y} \left[\frac{e^{-y}}{y}\right] dx\, dy$$

$$= \int_{y=0}^{\infty} \frac{e^{-y}}{y} \int_{x=0}^{y} dx\,dy$$

$$= \int_{y=0}^{\infty} \frac{e^{-y}}{y} [x]_0^y \, dy$$

$$= \int_{y=0}^{\infty} e^{-y} dy$$

$$= \left[\frac{e^{-y}}{-1} \right]_0^{\infty}$$

$$= [e^{-y}]_0^y$$

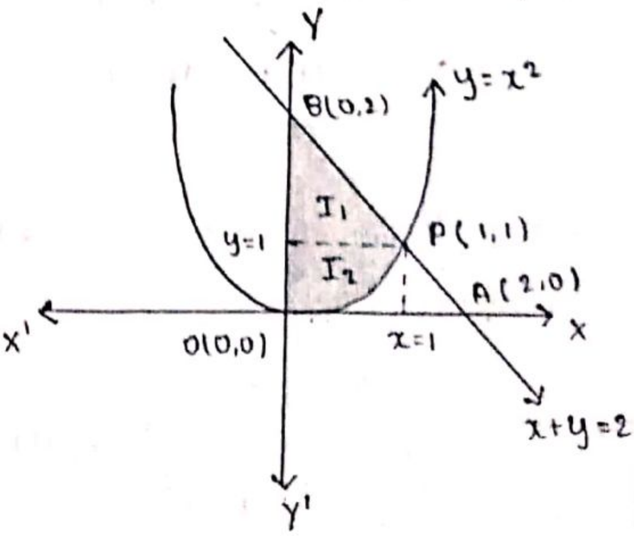
$$= -[e^{-\infty} - e^0]$$

$$= -[0-1]$$

$$= -(-1)$$

$$\Rightarrow I = 1$$

5. Evaluate $\int_0^1 \int_{x^2}^{2-x} xy \, dy \, dx$ using change of order of integration.



let $I = \int_{x=0}^{1} \int_{y=x^2}^{2-x} xy \, dy \, dx$

x varies from 0 to 1 and

y varies from $y = 2 - x$

$\Rightarrow y + x = 2$

or

$x + y = 2$

$x + y = 2$

$\frac{x}{2} + \frac{y}{2} = 2$

By change of order of integration the boundary region written as in

$$I = \int_{y=0}^{1} \int_{x=0}^{\sqrt{y}} xy \, dx \, dy + \int_{y=1}^{2} \int_{x=0}^{2-y} xy \, dx \, dy$$

$$I = I_1 + I_2$$

$$I_1 = \int_{y=0}^{1} \int_{x=0}^{\sqrt{y}} xy \cdot dx \, dy$$

$$= \int_{y=0}^{1} y \left[\frac{x^2}{2} \right]_0^{\sqrt{y}} dy$$

$$= \frac{1}{2} \int_0^1 y^2 \, dy$$

$$= \frac{1}{2} \left[\frac{y^3}{3} \right]_0^1$$

$$= \frac{1}{2} \left[\frac{1}{3} - 0 \right]$$

$$I_1 = \frac{1}{6}$$

$$I_2 = \int_{y=1}^{2} \int_{x=0}^{2-y} xy \, dx \, dy$$

$$= \int_{y=1}^{2} y \left[\frac{x^2}{2} \right]_0^{2-y} dy$$

$$= \frac{1}{2} \int_{y=0}^{2} y \, (2-y)^2 \, dy$$

$$= \frac{1}{2} \int_{y=1}^{y=1} y (y^2 - 4y + 4) dy$$

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$$= \frac{1}{2}\int_{y=1}^{2} (y^3 - 4y^2 + 4y)\, dy$$

$$= \frac{1}{2}\left[\frac{y^4}{4} - \frac{4y^3}{3} + \frac{4y^2}{2}\right]_1^2$$

$$= \frac{1}{2}\left\{\left[\frac{16}{4} - \frac{32}{3} + \frac{16}{2}\right] - \left[\frac{1}{4} - \frac{4}{3} + \frac{4}{2}\right]\right\}$$

$$= \frac{1}{2}\left[4 - \frac{32}{3} + 8 - \frac{1}{4} + \frac{4}{3} - 2\right]$$

$$= \frac{1}{2}\left[10 - \frac{28}{3} - \frac{1}{4}\right]$$

$$= \frac{1}{2}\left[\frac{120 - 1125 - 3}{12}\right]$$

$$= \frac{1}{2}\left[\frac{5}{12}\right]$$

$$\Rightarrow I_2 = \frac{5}{24}$$

$$I = I_1 + I_2$$

$$I = \frac{1}{6} + \frac{5}{24}$$

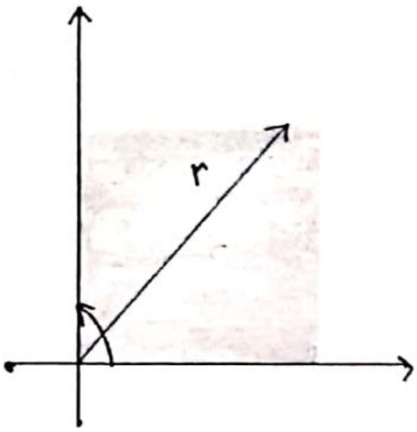
$$I = \frac{4+5}{24} = \frac{9}{24} = \frac{3}{8} \;//$$

6. Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$ by change into polar co-ordination.

Let $I = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy \longrightarrow (1)$

$$\Rightarrow I = \int_{x=0}^{\infty} \int_{y=0}^{\infty} e^{-(x^2+y^2)}\, dy\, dx$$

Let $x = r\cos\theta$, $y = r\sin\theta$



$\Rightarrow dx\,dy = r\,dr\,d\theta$

$$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2}\, r\,dr\,d\theta \rightarrow (2)$$

Let $r^2 = t$

$\Rightarrow 2r\,dr = dt$

$\Rightarrow r\,dr = \frac{1}{2}\,dt$

$\therefore$ UL : $r = \infty \Rightarrow t = \infty$

LL : $r = 0 \Rightarrow t = 0$

$$I = \int_{\theta=0}^{\pi/2} \int_{t=0}^{\infty} e^{-t}\,\frac{1}{2}\,dt\,d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi/2} \int_{t=0}^{\infty} e^{-t}\,dt\,.d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi/2} \left[\frac{e^{-t}}{-1}\right]_{t=0}^{\infty} d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi/2} \left[e^{-t}\right]_{0}^{\infty} d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi/2} \left[e^{-\infty} - e^{0}\right] d\theta$$

$$= \frac{1}{2}\int_{\theta=0}^{\pi/2} (0-1)\,d\theta$$

$$= \frac{1}{2} \int_{\theta=0}^{\pi/2} 1 \, d\theta$$

$$= \frac{1}{2} [\theta]_{0}^{\pi/2}$$

$$= \frac{1}{2} \left[\frac{\pi}{2} - 0 \right]$$

$$\Rightarrow I = \frac{\pi}{4}$$

7. Evaluate $\int_{x=0}^{\infty} \int_{y=0}^{\infty} e^{-(x^2+y^2)} \, dy \, dx = \frac{\pi}{4}$

$$I = \int_{x=0}^{\infty} \int_{y=0}^{\infty} e^{-x^2} \cdot e^{-y^2} \, dy \, dx = \frac{\pi}{4} \rightarrow ②$$

$$= \int_{x=0}^{\infty} e^{-x^2} dx \int_{y=0}^{\infty} e^{-y^2} dy = \frac{\pi}{4} \rightarrow ③$$

Let $x = y$

$$③ \Rightarrow \int_{x=0}^{\infty} e^{-x^2} dx \int_{x=0}^{\infty} e^{-x^2} dx = \frac{\pi}{4}$$

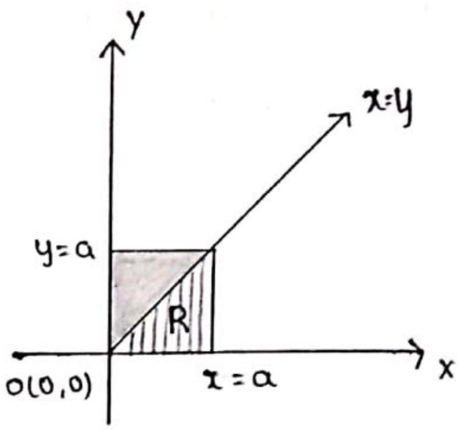
$$\Rightarrow \left[\int_{x=0}^{\infty} e^{-x^2} dx \right]^2 = \frac{\pi}{4}$$

$$\Rightarrow \int_{0}^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

8. Evaluate $\int_{0}^{a} \int_{y}^{a} \frac{x}{x^2+y^2} \, dx \, dy$

$$I = \int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy$$

$$I = \int_{y=0}^{a} \int_{x=y}^{a} \frac{x}{x^2+y^2}\, dx\, dy$$



$\therefore$ By change the order of integration, we have.

$$I = \int_{x=0}^{a} \int_{y=0}^{x} \frac{x}{x^2+y^2}\, dy\, dx$$

$$= \int_{x=0}^{a} x \int_{y=0}^{x} \frac{1}{x^2+y^2}\, dy\, dx$$

$$= \int_{x=0}^{a} x \frac{1}{x} \left[\tan^{-1}(y/x) \right]_{y=0}^{x} dx$$

$$= \int_{x=0}^{a} \left[\tan^{-1}(y/x) \right]_{y=0}^{x} dx$$

$$= \int_{x=0}^{a} \left[\pi/4 - 0 \right] dx$$

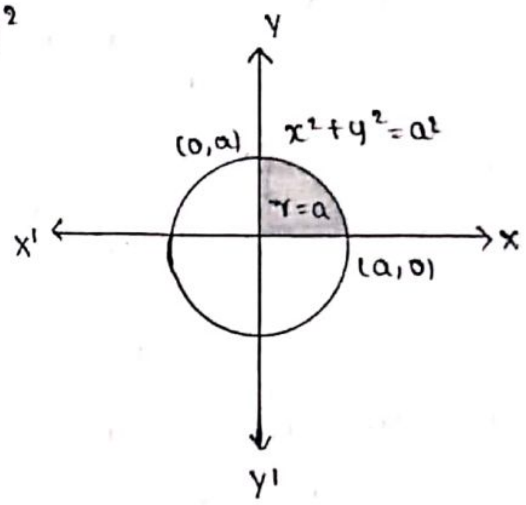
$$= \frac{\pi}{4} \int_{x=0}^{a} 1\, dx$$

$$= \frac{\pi}{4} [x]_0^a$$

$$I = \frac{\pi a}{4}$$

9. Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} y\sqrt{x^2+y^2}$ by change of polar coordinates

Here x varies from 0 to $\sqrt{a^2-y^2}$



$\Rightarrow x = \sqrt{a^2-y^2}$

$x^2 = a^2-y^2$

$x^2+y^2=a^2$

y varies from 0 to a

let $x = r\cos\theta$, $y = r\sin\theta$

$\Rightarrow dx\,dy = r\,dr\,d\theta$

$$I = \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r\sin\theta \cdot r \cdot r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r^3 \sin\theta\,dr\,d\theta$$

$$\Rightarrow I = \int_{\theta=0}^{\pi/2}\sin\theta\,d\theta \int_{r=\theta}^{a} r^3\,dr$$

$$= -\left[\cos\theta\right]_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^a$$

$$= -\left[\left[\cos\pi/2\right] - \cos\theta\right]\left[\frac{a^4}{4}\right]$$

$$= (0-1)\frac{a^4}{4}$$

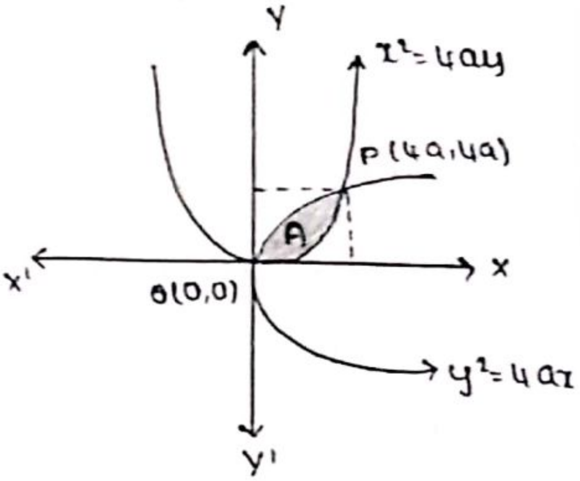
$$I = \frac{a^4}{4}$$

APPLICATIONS

1. Find the area between the parabolas. $y^2=4ax$, $x^2=4ay$

The area between the given two parabolas $y^2=4ax$, $x^2=4ay$ can be evaluated as

$$A = \iint dx\, dy$$



$$A = \int_{x=0}^{4a} \int_{y=\frac{x^2}{4a}}^{2\sqrt{ax}}$$

$$= \int_{x=0}^{4a} (y) \int_{x^2/4a}^{2\sqrt{ax}} dx$$

$$= \int_{0}^{4a} \left[2\sqrt{ax} - \frac{x^2}{4a} \right] dx$$

$$= 2\sqrt{a} \int_{0}^{4a} \sqrt{x}\, dx - \frac{1}{4a} \int_{0} x^2\, dx$$

$$= 2\sqrt{a} \int_{0}^{4a} \sqrt{x}\, dx - \frac{1}{4a} \int_{0}^{4a} x^2\, dx$$

$$= 2\sqrt{a} \left[\frac{x^{3/2}}{3/2} \right]_0^{4a} - \frac{1}{4a} \left[\frac{x^3}{3} \right]_0^{4a}$$

$$= \frac{4\sqrt{a}}{3} \left[x^{3/2} \right]_0^{4a} - \frac{1}{12a} \left[x^3 \right]_0^{4a}$$

$$= \frac{4\sqrt{a}}{3} (4a)^{3/2} - \frac{1}{12a} (4a)^3$$

$$= \frac{4\sqrt{a}}{3} (4a) 2\sqrt{a} \cdot \frac{1}{12a} 4a \times 4a^2$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3}$$

$$\Rightarrow A = \frac{16a^2}{3} \text{ sq units.}$$

$$\Rightarrow \int_{0}^{a} \int_{0}^{\sqrt{a^2 - y^2}} (x^2 + y^2)\, dx\, dy$$

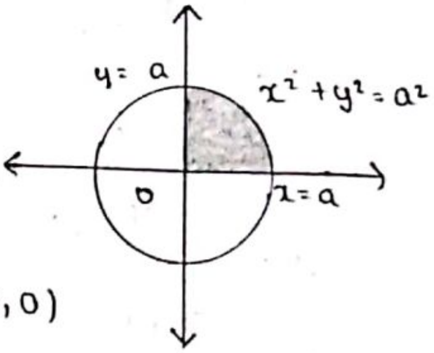
$$I = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} (x^2+y^2)\, dx\, dy \Rightarrow ①$$

x varies from $x=0$ to $x=\sqrt{a^2-y^2}$

$x = \sqrt{a^2-y^2}$

$x^2 = a^2-y^2$

$x^2+y^2 = a^2$



It is a circle having centre (0,0) will radius 'a' lies from 0 to a

let $x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

$$I = \int_{\theta=0}^{\pi/2} \int_{0}^{a} r^2(\cos^2\theta + \sin^2\theta)\, r\, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \int_{0}^{a} r^2\, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \left[\frac{r^4}{4}\right]_0^a d\theta$$

$$= \frac{1}{4} \int_{\theta=0}^{\pi/2} a^4\, d\theta$$

$$= \frac{a^4}{4} \int_{0}^{\pi/2} 1.\, d\theta$$

$$= \frac{a^4}{4} \int_{0}^{\pi/2} 1.\, d\theta$$

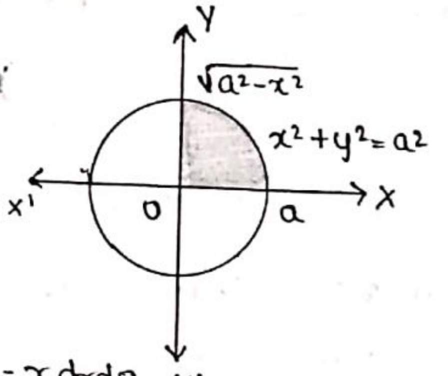
$$= \frac{a^4}{4} [1]_0^{\pi/2}$$

$$I = \frac{a^4}{4}\left[\frac{\pi}{2} - 0\right]$$

$$I = \frac{\pi a^4}{8}$$

2. Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} y^2\sqrt{x^2+y^2}\, dx\, dy$

$$I = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} y^2\sqrt{x^2+y^2}\, dx\, dy$$



x varies from $x=0$, to $x=a$

y varies from $y=0$ to $\sqrt{a^2-x^2}$

Let $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$

$$I = \int_{\theta=0}^{\pi/2} \int_0^a y^2\sqrt{x^2+y^2}\, dx\, dy$$

$$= \int_{\theta=0}^{\pi/2} \int_0^a r^2\sin^2\theta\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}\,.\, r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2} \int_0^a r^2\sin^2\theta\,.\, r\,.\, r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2} \int_0^a r^4\sin^2\theta\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^2\theta \int_0^a r^2\, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi/2} -2\sin\theta\cos\theta\left[\frac{r^5}{5}\right]_0^a d\theta$$

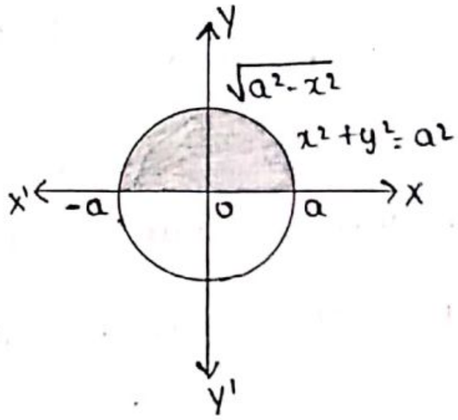
$$= \left[\frac{r^5}{5}\right]_0^a \left[\frac{2-1}{2} - \frac{\pi}{2}\right]$$

$$= \left[\frac{a^5}{5} - 0\right]\left[\frac{1}{2} \cdot \frac{\pi}{2}\right]$$

$$I = \frac{\pi a^5}{20} \text{ sq. units}$$

3. Evaluate change of order of integral $\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\,dy\,dx$

Let, $I = \int_{x=-a}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\ dy\,dx$



Let $x = r\cos\theta$, $y = r\sin\theta$

then x varies from $x = 0$ to a

y varies from $y=0$ to $y = \sqrt{a^2-x^2}$

$\Rightarrow y = \sqrt{a^2-x^2}$

$x^2+y^2 = a^2$

$$I = \int_0^{\pi}\int_0^{a} \sqrt{x^2+y^2}\ dy\,dx$$

$$= \int_0^{\pi}\int_0^{a} \sqrt{r^2\sin^2\theta + r^2\cos^2\theta}\ r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi}\int_0^{a} r\ r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^{a} r^2\,dr\ d\theta$$

$$= \int_0^{\pi}\left[\frac{r^2}{3}\right]_0^a d\theta$$

$$= \frac{a^3}{3}\left[\pi - 0\right]$$

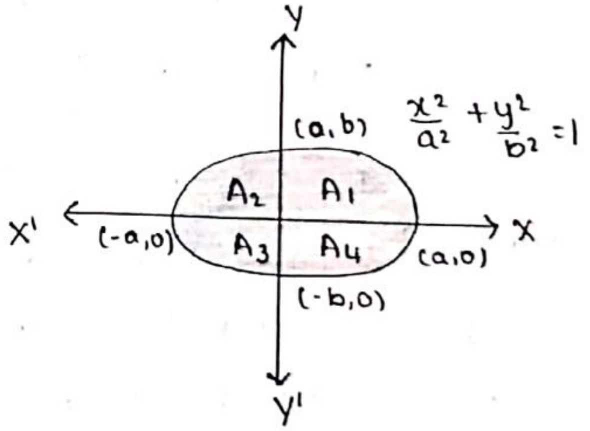
$$I = \frac{\pi a^3}{3}$$

④. Use double integration to find the area of a ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and hence find the area of the circle $\frac{x^2}{a^2} + \frac{y^2}{b^2} = a^2$.

$\Rightarrow$ Given the ellipse $x^2 + y^2 = a^2 \rightarrow$ ①

Covered the area with the co-ordinate axis as shown in the diagram.

$\therefore$ The area of the ellipse is



$A = A_1 4$

$A = 4\iint dx\, dy$

$\Rightarrow A = 4\int_{x=0}^{a}\int_{y=0}^{b/a\sqrt{a^2-x^2}} 1 .\, dy\, dx$

$A = 4\int_0^a [y]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$

$= 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\, dx$

$A = \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\, dx \rightarrow$ ②

Let $x = a\sin\theta \Rightarrow \theta = \sin^{-1}(x/a)$

$\Rightarrow dx = a\cos\theta\, d\theta$

U.L : $x = a \Rightarrow \theta = \frac{\pi}{2}$

L.L : $x = 0 \Rightarrow \theta = 0$

$\therefore A = \frac{4b}{a}\int_0^{\pi/2} \sqrt{a^2 - a^2\sin^2\theta}\; a\cos\theta\, d\theta$

$= \frac{4b}{a}\int_0^{\pi/2} a\sqrt{1-\sin^2\theta} \,.\, a\cos\theta\, d\theta$

$$= 4b \int_0^{\pi/2} a \cos^2\theta \, d\theta$$

$$= 4ab \int_0^{\pi/2} \left[\frac{1+\cos\theta}{2} \right] d\theta$$

$$= 2ab \int_0^{\pi/2} (1+\cos 2\theta) \, d\theta$$

$$= 2ab \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/2}$$

$$= 2ab \left[\pi/2 + 0 \right]$$

$$\Rightarrow A = 2ab \left(\frac{\pi}{2} \right)$$

$$A = \pi ab \text{ Sq units}$$

when $b=a$, then the given ellipse becomes a Circle $x^2+y^2=a^2$ and its area is

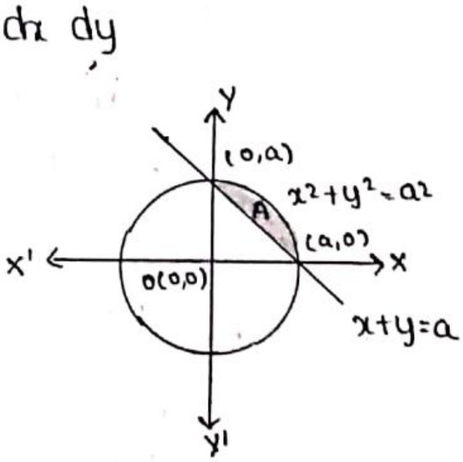
$$A = \pi \times a \times a = \pi a^2 \text{ Sq units}$$

5. Find the area by double integration between the circle $x^2+y^2=a^2$ and to the straight line $x+y=a$

→ Given that the straight line cuts the x axis at $(a,0)$ and y = axis at $(0,a)$

∴ The area between the circle $x^2+y^2=a^2$ and the straight line $x+y=a$ is $A=\iint dx\,dy$

$$A = \int_0^a \int_{y=a-x}^{\sqrt{a^2-x^2}} 1 \, dy \, dx$$



$$= \int_0^a \left[y \right]_{y=a-x}^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \left(\sqrt{a^2-x^2} - (a-x) \right) dx$$

$$A = \int_{x=0}^{a} \sqrt{a^2-x^2}\, dx - \int_{0}^{a} (a-x)\, dx$$

let $x = a \sin\theta \Rightarrow \theta = \sin^{-1}(x/a)$

$= dx = a \cos\theta$

UL : $x = a \Rightarrow \theta = \frac{\pi}{2}$

LL : $x = 0 \Rightarrow \theta = 0$

$$\therefore A = \int_{0}^{\pi/2} \sqrt{a^2 - a^2\sin^2\theta}\,.\, a\cos\theta\, d\theta - \left[ax - \frac{x^2}{2}\right]_0^a$$

$$= \int_{0}^{\pi/2} a^2\cos^2\theta\, d\theta - \left\{\left(a^2 - \frac{a^2}{2}\right) - (0-0)\right\}$$

$$= a^2 \int_{0}^{\pi/2} \cos^2\theta\, d\theta - \frac{a^2}{2}$$

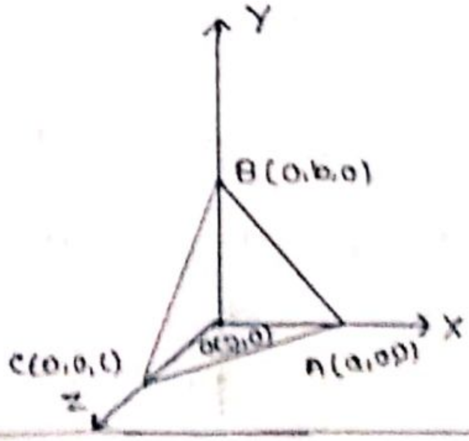
$$= a^2 \left(\frac{2-1}{2}\right) \frac{\pi}{2} - \frac{a^2}{2}$$

$$A = \frac{\pi a^2}{4} - \frac{a^2}{2} \text{ sq units.}$$

$\Rightarrow$ <u>VOLUME OF TETRAHYDRAL</u>

1. Find the volume of tetrahydral bounded by the planes $x=0$, $y=0$, $z=0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$\Rightarrow$ Given tetrahydral bounded by the plan $x=0$, $y=0$, $z=0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$



$\therefore$ The volume $V = \iiint dv$

$$= \int_{x=}^{a} \int_{y=0}^{b/a(a-x)} \int_{z=0}^{c(1-x/a-y/b)} 1\, dz\, dy\, dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{b/a(a-x)} [z]_0^{c[1-\frac{x}{a}-\frac{y}{b}]} dy\, dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{b/a(a-x)} c\left[1-\frac{x}{a}-\frac{y}{b}\right] dy\, dx$$

$$= c\int_{x=0}^{a} \left[y-\frac{xy}{a}-\frac{y^2}{2b}\right]_{y=0}^{b/a(a-x)} dx$$

$$= c\int_{x=0}^{a} \left[\frac{b}{a}(a-x)-\frac{x}{a}\cdot\frac{b}{a}(a-x)-\frac{1}{2b}\,\frac{b^2}{a^2}(a-x^2)\right]dx$$

$$= c\int_{0}^{a} \left[\frac{b}{a}(a-x)-\frac{b}{a^2}x(a-x)-\frac{b}{2a^2}(a-x^2)\right]dx$$

$$= c\int_{0}^{a} \frac{b}{a^2}(a-x)(a-x)-\frac{1}{2}(a-x)]\,dx$$

$$= c\int_{0}^{a} \frac{b}{a^2}(a-x)^2\left[1-\frac{1}{2}\right]dx$$

$$= c\int_{0}^{a} \frac{b}{2a^2}(a-x^2)\,dx$$

$$= \frac{bc}{2a^2}\int_{0}^{a}(x^2-2ax+a^2)\,dx$$

$$= \frac{bc}{2a^2}\left[\frac{x^3}{3}-ax^2+a^2x\right]_0^a$$

$$= \frac{bc}{2a^2}\left[\frac{a^3}{3}-a^3+a^3\right]$$

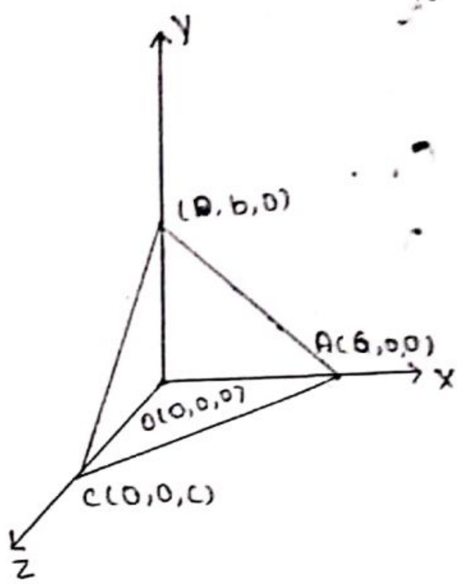
$$\therefore V = \frac{bc}{2a^2}\times\frac{a^3}{3}$$

$$\Rightarrow V = \frac{abc}{6} \text{ cubic meter.}$$

2. Find the volume of the tetrahydral bounded by the planes, $x=0$, $y=0$, $z=0$ and $x+2y+3z=6$.

$\Rightarrow$ Given that the volume of tetrahydral bounded by plans

$x=0,\ y=0,\ z=0$ and $x+2y+3z=6$



$$V=\iiint 1\,dV$$

$$= \int_{x=0}^{6}\int_{y=0}^{1/2(6-x)}\int_{z=0}^{1/3(6-x-2y)} 1 .\, dz\, dy\, dx$$

$$= \int_{x=0}^{6}\int_{y=0}^{1/2(6-x)} [z]_{(1)}^{1/3(6-x-2y)}$$

$$= \frac{1}{3}\int_{x=0}^{6} [6y-xy-2y]\, dy\, dx$$

$$= \frac{1}{3}\int_{x=0}^{6} [6x-xy-y^2]_{y=0}^{1/2(6-x)}$$

$$= \frac{1}{3}\int_{0}^{6} (6-x)\, y-y^2 \Big|_{y=0}^{1/2(6-x)} dx$$

$$= \frac{1}{3}\int_{0}^{6}\left[(6-x)\frac{(6-x)}{2} - \frac{(6-x^2)^2}{4}\right] dx$$

$$= \frac{1}{3}\int_{0}^{6}\left[\frac{(6-x)^2}{2} - \frac{(6-x)^2}{4}\right] dx$$

$$= \frac{1}{3}\int_{0}^{6}\left(\frac{1}{2}-\frac{1}{4}\right)(6-x)^2\, dx$$

$$= \frac{1}{3}\int_{0}^{6}\frac{1}{4}(6-x)^2\, dx$$

$$= \frac{1}{12}\int_{0}^{6}(x^2-12x+36)\, dx$$

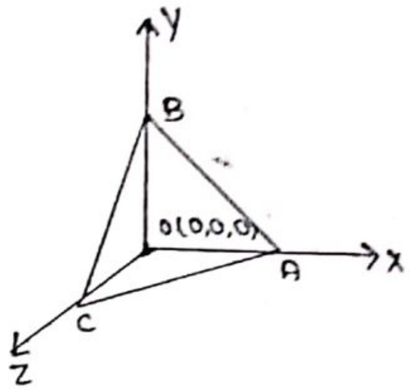
$$= \frac{1}{12}\left[\frac{x^3}{3}-6x^2+36x\right]_0^6$$

$$= \frac{1}{12}\left[\frac{216}{3}-216+216\right] = \frac{216}{36} \Rightarrow V = 6 \text{ cubic units}$$

3. Find the volume of a tetrahydral bounded by the plane $x=0$, $y=0$, $z=0$ and $x+y+z=1$

Given that, the volume of the tetrahydral

$x=0$, $y=0$, $z=0$ and $x+y+z=1$



$$= \int_{x=0}^{1} \int_{y=0}^{(1-x)} \int_{z=0}^{(1-x-y)} 1 \, dz \, dy \, dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{(x-1)} [z]_0^{(1-x-y)} \, dy \, dx$$

$$= \int_{x=0}^{1} \int_{0}^{(1-x)} (1-x-y) \, dy \, dx$$

$$= \int_{a}^{1} \left[y - xy - \frac{y^2}{2} \right]_0^{(1-x)}$$

$$= \int_{0}^{1} \left[(1-x) - x(1-x) - \frac{(1-x)^2}{2} \right]^{1} dx$$

$$= \frac{1}{2} \int_{0}^{1} (1-x) \left[(1-x) - \frac{1}{2}(1-x) \right] dx$$

$$= \int_{0}^{1} (1-x)^2 \left[1 - \frac{1}{2} \right] dx$$

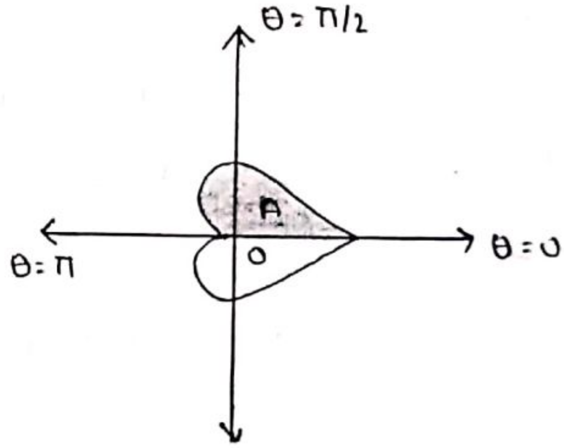
$$= \frac{1}{2} \int_{0}^{1} (1 - 2x + x^2) \, dx$$

$$= \frac{1}{2} \left[x - \frac{2x^2}{2} + \frac{x^3}{3} \right]_0^1$$

$$= \frac{1}{2} \left[1 - \frac{2(1)^2}{2} + \frac{1^3}{3} \right]$$

$$= \frac{1}{2} \left[1 - 0 - \frac{1}{3} \right] = \frac{1}{6} \text{ cubic units.}$$

4. Find the area bounded by $\theta = 0$, $\theta = \pi$ of a Cardiac $r = a(1+\cos\theta)$



$$A = \iint dx\,dy$$

$$= \iint r\,dr\,d\theta$$

$$= \int_0^{\pi} \int_{r=0}^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi} \left[r^2/2 \right]_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{1}{2} \int_{\theta=0}^{\pi} a^2 (1+\cos\theta)^2\, d\theta$$

$$= \frac{a^2}{2} \int_{\theta=0}^{\pi} (1+\cos\theta)^2\, d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi} \left[1 + 2\cos\theta + \cos^2\theta \right] d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi} \left[1 + 2\cos\theta + \frac{1+\cos 2\theta}{2} \right] d\theta$$

$$= \frac{a^2}{4} \int_0^{\pi} \left[2 + 4\cos\theta + 1 + \cos 2\theta \right] d\theta$$

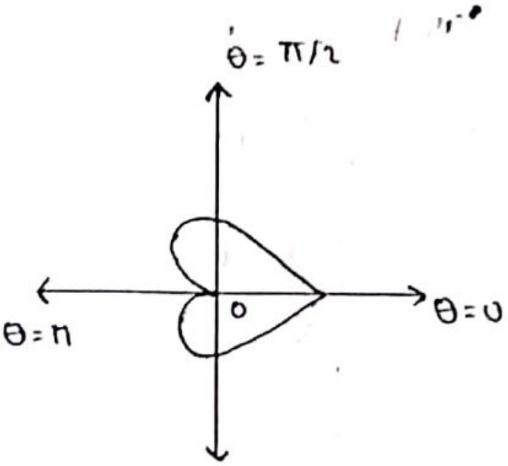
$$= \frac{a^2}{4} \int_0^{\pi} \left[\cos 2\theta + 4\cos\theta + 3 \right] d\theta$$

$$= \frac{a^2}{4} \left[\frac{\sin 2\theta}{2} + 4\sin\theta + 3\theta \right]_0^{\pi}$$

$$= \frac{a^2}{4} \left[(0 + 0 + 3\pi) - (0 + 0 + 0) \right]$$

$$A = \frac{3a^2\pi}{4} \text{ Sq units}$$

5. Find the volume of the cardiac $r = a(1+\cos\theta)$ above the intial line



W.K.T

The volume of the cardiac $r = a(1+\cos\theta)$ above the intial line

$$V = \iint 2\pi r^2 \sin\theta \, dr \, d\theta$$

$$= \int_{\theta=0}^{\pi} \int_{r=0}^{a(1+\cos\theta)} 2\pi r^2 \sin\theta \, dr \, d\theta$$

$$= 2\pi \int_{\theta=0}^{\pi} \sin\theta \left[\frac{r^3}{3}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{2\pi}{3} \int_{\theta}^{\pi} a^3 (1+\cos\theta)^3 \sin\theta \, d\theta$$

$$= \frac{2\pi a^3}{3} \int_{\theta=0}^{\pi} (1+\cos\theta)^3 \sin\theta \, d\theta$$

Let $1+\cos\theta = t$

$\Rightarrow -\sin\theta \, d\theta = dt$

$\Rightarrow \sin\theta \, d\theta = -dt$

UL : $\theta = \pi \Rightarrow t = 1+\cos(\pi) = 1-1 = 0$

LL : $\theta = 0 \Rightarrow t = 1+\cos\theta = 1+1 = 2$

$$V = \frac{2\pi a^3}{3} \int_2^0 t^3(-dt)$$

$$= \frac{2\pi a^3}{3} \int_0^2 t^3 dt$$

$$= \frac{2\pi a^3}{3} \left[\frac{t^4}{4}\right]_0^2$$

$$= \frac{2\pi a^3}{4\times 3} \times (2^4 - 0)$$

$$= \frac{2\pi a^3}{4\times 3} \times 2\times 2\times 2\times 2$$

$$A = \frac{8\pi a^3}{3} \text{ cubic units}$$

⇒ **Beta - Gamma functions**

Defination :- For any $(m,n)>0$, then the improper integral can be defined as $\beta(m,n) \int_1^0 x^{n-1}(1-x)^{n-1}\,dx \rightarrow (1)$ is called the Beta function in m and n.

Let, $x = \sin^2\theta \Rightarrow \theta = \sin^{-1}(\sqrt{x})$

$\Rightarrow dx = 2\sin\theta\cos\theta\, d\theta$

UL : $x=1 \Rightarrow \theta = \frac{\pi}{2}$

LL : $x=0 \Rightarrow \theta = 0$

(1) $\Rightarrow \beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1} . (1-\sin^2\theta)^{n-1}\, 2\sin\theta\cos\theta\, d\theta$

$= 2\int_0^{\pi/2} \sin^{2(m-1)}\theta . \cos^{2(m-1)}\theta . \sin\theta\cos\theta\, d\theta$

$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-2+1}_\theta . \cos^{2n-2+1}_\theta\, d\theta$

$\Rightarrow \beta(m,n) = 2\int_0^{\pi/2} \sin\theta^{2m-1} . \cos\theta^{2n-1}\, d\theta \rightarrow (2)$

let $p = 2m-1$, $q = 2n-1$

$\Rightarrow m = \frac{p+1}{2}$, $n = \frac{q+1}{2}$

(2) $\Rightarrow \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\pi/2} \sin^p\theta . \cos^q\theta\, d\theta$

$$\Rightarrow \int_0^{\pi/2} \sin^p\theta \cos^q d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

Gamma function: for every $n>0$ gamma function can be defined as

$$\Gamma n = \int_0^{\infty} e^{-x} x^{n-1} dx \rightarrow (1)$$

let $x = y^2$

$$\Rightarrow dx = 2y\, dy$$

UL : $x = \infty \Rightarrow y = \infty$

LL : $x = 0 \Rightarrow y = 0$

$$\therefore (1) \Rightarrow \Gamma n = \int_0^{\infty} e^{-y^2} (y^2)^{n-1} 2y\, dy$$

$$= 2\int_0^{\infty} e^{-y^2} y^{2n-2+1} dy$$

$$\Rightarrow \Gamma n = 2\int_0^{\infty} e^{-y^2} y^{2n-1} dy$$

$$\Rightarrow \Gamma n = 2\int_0^{\infty} e^{-x^2} x^{2n-1} dx \quad (\because y = x)$$

RELATION BETWEEN BETA AND GAMMA FUNCTION

we know that,

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2n-1}\theta \cos^{2m-1}\theta\, d\theta \rightarrow (1)$$

$$\Gamma m = 2\int_0^{\infty} e^{-x^2} x^{2m-1} dx \rightarrow (2)$$

$$\Gamma n = 2\int_0^{\infty} e^{-y^2} y^{2n-1} dy \rightarrow (3)$$

$$\Gamma(m+n) = 2\int_0^\infty e^{-r^2} r^{2(m+n)-1}\, dr \rightarrow (4)$$

Let $\Gamma m\, \Gamma n = 2\int_0^\infty e^{-x^2} x^{2m-1} y^{2n-1}\, dx\, dy \rightarrow (5)$

Let $x = r\cos\theta$, $y = r\sin\theta$

$\Rightarrow dx\, dy = r\, dr\, d\theta$

(5) $\Rightarrow \Gamma m\, \Gamma n = 4\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2} (r\cos\theta)^{2m-1} (r\sin\theta)^{2n-1}\, r\, dr\, d\theta$

$$= 4\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2} r^{2m-1+2n-1+1} \sin^{2n-1}\theta \cos^{2m-1}\theta\, dr\, d\theta$$

$$= 4\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2} r^{2(m+n)-1} \sin^{2n-1}\theta \cos^{2m-1}\theta\ \theta\, dr\, d\theta$$

$$= 4\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2} r^{2(m+n)-1} \sin^{2n-1}\theta \cos^{2n-1}\theta\, dr\, d\theta$$

$$\Gamma m\, \Gamma n = \left[2\int_0^\infty e^{-r^2} r^{2(m+n)-1}\, dr\right]\cdot\left[2\int_0^{\pi/2} \sin^{2n-1}\theta \cos^{2n-1}\theta\, d\theta\right]$$

$$\Gamma m\, \Gamma n = (\Gamma(m+n))\, \beta(m,n)$$

$$\beta(m,n) = \frac{\Gamma m\, \Gamma n}{\Gamma(m+n)} \quad //$$

Note:-

1. $\beta(m,n) = \beta(n,m)$
2. $\Gamma(n+1) = n\Gamma n = n!$
3. $\Gamma 1 = 1$
4. $\Gamma(1/2) = \sqrt{\pi}$
5. $\Gamma n\, \Gamma(1-n) = \dfrac{\pi}{\sin(n\pi)}$

① Show that $\Gamma 1/2 = \sqrt{\pi}$ using Beta - gamma function.

WKT

$$\Gamma m = 2\int_0^\infty e^{-x^2} x^{2m-1}\, dx \rightarrow ①$$

$$\Gamma n = 2\int_0^\infty e^{-y^2} y^{2n-1}\, dy \rightarrow ②$$

$$\therefore \Gamma m\, \Gamma n = \left[2\int_0^\infty e^{-x^2} x^{2n-1}\, dx\right]\left[2\int_0^\infty e^{-y^2} y^{2n-1}\, dy\right]$$

$$\Rightarrow \Gamma m\, \Gamma n = \int_0^\infty \int_0^\infty e^{-x^2} \cdot e^{-y^2} x^{2m-1} y^{2n-1}\, dx\, dy \rightarrow ③$$

Let $m = \frac{1}{2}$, $n = \frac{1}{2}$

$$② \Rightarrow \Gamma 1/2\, \Gamma 1/2 = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} x^0 y^0\, dx\, dy$$

$$\Rightarrow (\Gamma 1/2)^2 = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy \rightarrow ④$$

Let $x = r\cos\theta$, $y = r\sin\theta$

$\Rightarrow dx\,dy = r\, dr\, d\theta$

$$\therefore ④\ (\Gamma 1/2)^2 = 4\int_{r=0}^{\infty} \int_{\theta=0}^{\pi/2} e^{-r^2} r\, dr\, d\theta$$

$$= 4\int_{r=0}^{\infty} r e^{-r^2}\, dr\, [\theta]_0^{\pi/2}$$

$$= \frac{4\pi}{2}\int_{r=0}^{\infty} r\, e^{-r^2}\, dr$$

$$= \pi\int_0^\infty (2r) e^{-r^2}\, dr$$

Let $r^2 = t$

$\Rightarrow 2r\, dr = dt$

$$(\Gamma 1/2)^2 = \pi\int_0^\infty e^{-t}\, dt$$

$$\left(\Gamma\tfrac{1}{2}\right)^2 = \pi\left(\frac{e^{-t}}{-1}\right)_0^{\infty}$$

$$= -\pi\left[e^{-t}\right]_0^{\infty}$$

$$= \pi\left[e^{-\infty}-1\right]$$

$$= -\pi(0-1)$$

$$\Rightarrow \left(\Gamma\tfrac{1}{2}\right)^2 = \pi$$

$$\Gamma\tfrac{1}{2} = \sqrt{\pi}$$ //

2. Show that $\int_0^2 (4-x^2)^{3/2}\,dx = 3\pi$

$$I = \int_0^2 (4-x^2)^{3/2}\,dx$$

Let $x = 2\sin\theta \Rightarrow \theta = \sin^{-1}(x/2)$.

$$\Rightarrow dx = 2\cos\theta\,d\theta$$

UL : $x = 2 \Rightarrow \theta = \sin^{-1}(1) = \frac{\pi}{2}$

LL : $x = 0 \Rightarrow \theta = \sin^{-1}(0) = 0$

$$\therefore I = \int_0^{\pi/2} (4-4\sin^2\theta)^{3/2} \cdot 2\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} (4)^{3/2}(\cos^2\theta)^{3/2}\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} (2^2)^{3/2}\cos^3\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} (2^2)^{3/2}\cos^3\theta\cos\theta\,d\theta$$

$$I = 16\int_0^{\pi/2}\cos^4\theta\,d\theta$$

$$\therefore \int_0^{\pi/2}\cos^4\theta\,d\theta = \int_0^{\pi/2}\sin^0\theta\cos^4\theta = \int_0^{\pi/2}\sin^0\theta\cos^4\theta\,d\theta$$

$\therefore p = 0,\ q = 4$

$$\int_0^{\pi/2} \cos^4\theta\, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$= \frac{1}{2}\beta\left(\frac{1}{2}, \frac{5}{2}\right)$$

$$= \frac{1}{2}\,\frac{\Gamma 1/2\ \Gamma 5/2}{\Gamma\, 1/2 + 5/2}$$

$$= \frac{\frac{1}{2}\Gamma 1/2\ (5/2 - 1)\ \Gamma 5/2 - 1}{\Gamma 3}$$

$$= \frac{\frac{1}{2}\Gamma 1/2\ (3/2)\ \Gamma 3/2}{2!}$$

$$= \frac{\frac{1}{2}\Gamma 1/2\ (3/2)\ (3/2 - 1)\ \Gamma 3/2 - 1}{2}$$

$$= \frac{\frac{1}{2}\Gamma 1/2\ (3/2)\ (3/2 - 1)\ \Gamma 3/2 - 1}{2}$$

$$= \frac{3}{16}\sqrt{\pi}\ \sqrt{\pi}$$

$$\int_0^{\pi/2} \cos^4\theta\, d\theta = \frac{3\pi}{6}$$

$$\therefore I = 16 \times \frac{3\pi}{16}$$

$$\Rightarrow I = 3\pi$$

3. Show that $\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \times \int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}}\, d\theta$

$I_1 \times I_2$

$$I_1 = \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta$$

$$= \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta$$

$p = 1/2 \, , \, q = 0$

$$I_1 = \frac{1}{2} \beta \left(\frac{p+1}{2} , \frac{q+1}{2} \right)$$

$$= \frac{1}{2} \beta \left(\frac{1/2+1}{2} , \frac{0+1}{2} \right)$$

$$= \frac{1}{2} \beta \left(3/4 \, ; \, 1/2 \right)$$

$$= \frac{1}{2} \frac{\Gamma 3/4 \; \Gamma 1/2}{\Gamma 3/4 + 1/2}$$

$$= \frac{1}{2} \frac{\Gamma 3/4 \; \Gamma 1/2}{\Gamma 5/4}$$

$$= \frac{1}{2} \frac{\Gamma 3/4 \; \Gamma 1/2}{\frac{1}{4} \Gamma 1/4}$$

$$\Rightarrow I_1 = \frac{2 \Gamma 3/4 \; \Gamma 1/2}{\Gamma 1/4}$$

$$\therefore I_2 = \int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}} \, d\theta$$

$$= \int_0^{\pi/2} \frac{1}{\sin^{1/2}\theta}$$

$$= \int_0^{\pi/2} \sin^{-1/2}\theta \cos^0\theta \, d\theta$$

$p = -1/2 \quad , q = 0$

$$I_2 = \frac{1}{2} \beta \left(\frac{-1/2+1}{2} , \frac{0+1}{2} \right)$$

$$= \frac{1}{2} \beta \left(1/4 \, , \, 1/2 \right)$$

$$= \frac{1}{2} \frac{\Gamma(1/4)\,\Gamma(1/2)}{\Gamma(1/4 + 1/2)}$$

$$\Rightarrow I_2 = 1/2 \frac{\Gamma(1/4)\,\Gamma(1/2)}{\Gamma(3/4)}$$

$$I = 2\frac{\Gamma(2/4)\,\Gamma(1/2)}{\Gamma(1/4)} \cdot 1/2 \frac{\Gamma(1/4)\,\Gamma(1/2)}{\Gamma(3/4)}$$

$$\therefore I = \Gamma(1/2)\,\Gamma(1/2)$$

$$I = \sqrt{\pi}\,\sqrt{\pi}$$

$$I = \pi$$

4. Evaluate $\int_0^1 x^{3/2}(1-x)^{1/2}\,dx$ by expressing interms of Beta - gamma function.

Let $I = \int_0^1 x^{3/2}(1-x)^{1/2}\,dx$

Let $x = \sin^2\theta \Rightarrow \theta = \sin^{-1}(\sqrt{x})$

$$dx = 2\sin\theta\cos\theta\,d\theta$$

UL : $x = 1 \Rightarrow \theta = \pi/2$

LL : $x = 0 \Rightarrow \theta = 0$

$$I = \int_0^{\pi/2} (\sin^3\theta)^{3/2}(1-\sin^2\theta)^{1/2}\, 2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^3\theta\,\cos\theta\,\sin\theta\,\cos\theta\,d\theta$$

$$I = \int_0^{\pi/2} \sin^4\theta\,\cos^2\theta\,d\theta$$

$$= \int_0^{\pi/2} \sin^4\theta\,\cos^2\theta\,d\theta \quad \cdot \int_0^{\pi/2} \sin^p\theta\,\cos^q\theta\,d\theta$$

$\therefore p = 4, \; q = 2$

$$\int_0^{\pi/2} \sin^4\theta \cos^2\theta \, d\theta = \frac{1}{2} \beta\left(\frac{5}{2}, \frac{3}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma 5/2 \; \Gamma 3/2}{\Gamma 4}$$

$$= \frac{1}{2} \frac{3/2 \cdot 1/2 \, \Gamma 1/2 \cdot 1/2 \, \Gamma 1/2}{3!}$$

$$= \frac{1}{2} \frac{3/2 \; 1/2 \, \Gamma 1/2 \;\; 1/2 \, \Gamma 1/2}{6}$$

$$= \frac{1}{2^5} \sqrt{\pi} \sqrt{\pi}$$

$$= \frac{\pi}{2^5}$$

$$\therefore I = \frac{\pi}{16}$$

5. Evaluate $\int_0^4 x^{3/2} (4-x)^{5/2} \, dx$ by using Beta-gamma function. Show that $\beta(m,n) = \int_0^\infty \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} \, dx$

$$\beta(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} \, dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} \, dx$$

Let $I = \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} \, dx$

then, $x = \frac{1}{y}$, $y = \frac{1}{x}$

$$dx = \frac{1}{y^2} \, dy$$

U.L : $x = \infty \Rightarrow y = 0$

LL : $x = 1 \Rightarrow y = 1$

$$\therefore I = \int_0^1 \frac{(1/y)^{m-1}}{(1+1/y)^{m+n}} \left(-\frac{1}{y^2}\right) dy$$

$$= \int_0^1 \frac{\frac{1}{y^{m-1}} \frac{1}{y^2}}{\frac{(y+1)^{m+n}}{y^{(m+n)}}} dy$$

$$= \int_0^1 \frac{\frac{1}{y^{m-1}}}{\frac{(1+y)^{m+n}}{y^{m+n}}} dy$$

$$= \int_0^1 \frac{y^{m+n}}{y^{m+1}(1+y)^{m+n}} dy$$

$$= \int_0^1 \frac{y^m y^n}{y^m y^1 (1+y)^{m+n}} dy$$

$$I = \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

$$I = \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

$$\therefore \beta(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx \quad //$$

6. Show that $\int_0^\infty \frac{e^{-x^2} dx}{\sqrt{x}} \times \int_0^\infty \sqrt{x}\, e^{-x^2} dx = \frac{\pi}{2\sqrt{2}}$

$$I = \int_0^\infty e^{-x^2} \sqrt{x}\, dx \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}} dx$$

$$\Rightarrow I = I_1 \times I_2$$

$$I_1 = \int_0^\infty e^{-x^2} \sqrt{x}\, dx \Rightarrow \int_0^\infty e^{-x^2} x^{1/2}\, dx$$

$$= \int_0^\infty e^{-x^2} x^{2n-1}\, dx$$

$$\therefore 2n - 1 = \frac{1}{2} \Rightarrow 2n = \frac{3}{2} \Rightarrow n = \frac{3}{4}$$

$$I_1 = \frac{\Gamma n}{2} = \frac{\Gamma 3/4}{2}$$

$$I_2 = \int_0^\infty e^{-x^2}/\sqrt{x}\, dx \Rightarrow \int_0^\infty e^{-x^2} x^{-1/2}\, dx \Rightarrow \int_0^\infty e^{-x^2} x^{2m-1}\, dx$$

$$\therefore 2m - 1 = -\frac{1}{2} \Rightarrow 2m = 1 - \frac{1}{2} \Rightarrow 2m = \frac{1}{2} \Rightarrow m = \frac{1}{4}$$

$$I_2 = \frac{\Gamma 1/4}{2}$$

$$I = I_1 \times I_2$$

$$I = \frac{\Gamma 3/4}{2} \times \frac{\Gamma 1/4}{2} \Rightarrow \frac{1}{4} \Gamma 3/4\, \Gamma 1/4 \Rightarrow \frac{1}{4} \Gamma 1 - 1/4 \quad \Gamma \frac{1}{4}$$

$$\Rightarrow \frac{1}{4} \frac{\pi}{\text{Sinn}\, \pi}$$

$$I = \frac{1}{4} \frac{\pi}{\sin \frac{\pi}{4}}$$

$$I = \frac{1}{4} \frac{\pi}{\frac{1}{\sqrt{2}}}$$

$$= \frac{\sqrt{2}}{4} \pi \Rightarrow \frac{\pi}{2\sqrt{2}} = \frac{\pi}{2\sqrt{2}}$$ hence proved.

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LECTURE NOTES

CALCULUS & LINEAR ALGEBRA (18MAT11)

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CALCULUS AND LINEAR ALGEBRA

MODULE-4

DIFFERENTIAL EQUATION

If an equation contain one dependant variable and then derivative with respect to one or more independent Variable is called Differential equation.

There are are two types of D. E

1. Ordinary differential equation
2. partial differential equation

=> Ordinary differential equation :- If an equation contain one dependant variable and its derivative with respect to one independant variable.

ex :- $x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} = 0$

=> Partial differential equation :- If an equation Contain one depeandant variable and derivatives with respect to two or more variable

ex :- $x \frac{dz}{dx} + y \frac{dz}{dy} = 2z$

⇒ Order and degree of differential equation

The highest derivative in a given D.E is called order it's power is called degree

ex :- $x^2 \left(\frac{d^3y}{dx^3}\right)^4 - 2x\left(\frac{d^2y}{dx^2}\right)^5 + \left(\frac{dy}{dx}\right)^5 + y = 0$

The highest derivative is $\frac{d^3y}{dx^3}$ and power 4

order of D.E = 3

degree of D.E = 4

⇒ Solution of 1st order and 1st degree D.E

Generally 1st order and 1st degree D.E is in the form $\frac{dy}{dx} = f(x,y)$

⇒ Exact Differential equation.

Step 1 :- write the given D.E is in the form $M(x,y)dx + N(x,y)dy = 0$

Step 2 :- Identify the M and N find $\frac{\partial M}{\partial y}$, $\frac{\partial N}{\partial y}$

Step 3 :- if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ we say that given D.E is exact D.E

Step 4 :- Write the Solution for E.D.E

$\int M(x,y)\, dx + \int$ (The terms which don't contain x in N) $dy = c$

Problems :-

1. Solve $\frac{dy}{dx} + \frac{2x+3y-1}{3x+4y-2} = 0$

$(2x+3y-1)\,dx + (3x+4y+2)dy = 0$

$M = (2x+3y-1)$ $\quad N = (3x+4y+2)$

$\frac{\partial M}{\partial y} = 3 \qquad \frac{\partial N}{\partial x} = 3$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 3$

The given D.E is E.DE

$\int (xy)\,dx + \int (\text{the term which don't contain } x)\,dy = c$

$\Rightarrow \int (2x + 3y - 1)\,dx + \int (4y + 2)\,dy = c$

$\Rightarrow \frac{2x^2}{2} + 3yx - x + \frac{4y^2}{2} + 2y = c$

$\Rightarrow x^2 + 3yx - x + 2y^2 + 2y = c$

$\Rightarrow x^2 + 2y^2 - x + 2y + 3yx = c$

02. Solve $(2x + y + 1)\,dx + (x + 2y + 1)\,dy = 0$

$(2x + y + 1)\,dx + (x + 2y + 1)\,dy = 0$

$M = (2x + y + 1) \qquad N = (x + 2y + 1)$

$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = 1$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 1$

$\therefore$ The DE is E.D.E

$\Rightarrow \int M(x,y)\,dx + \int (\text{The term which don't contain } x \text{ in } N)\,dy = c$

$\Rightarrow \int (2x + y + 1)\,dx + \int (2y + 1)\,dy = c$

$\Rightarrow x^2 + yx + x + y^2 + y = c$

3. Solve $(y^3 - 3x^2y)\,dx - (x^3 - 3xy^2)\,dy = 0$

$(y^3 - 3x^2y)\,dx + (-x^3 + 3xy^2)\,dy = 0$

$M = y^3 - 3x^2y \qquad N = 3xy^2 - x^3$

$\frac{\partial M}{\partial y} = 3y^2 - 3x^2 \qquad \frac{\partial N}{\partial x} = 3y^2 - 3x^2$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 3y^2 - 3x^2$

$\therefore$ The DE is E DE

$\Rightarrow \int (y^3 - 3x^2y)\,dx + \int 0 = c \Rightarrow xy^3 - x^3y = c$

$xy[x^2 - y^2] = c$

4. Solve $[5x^4 + 3x^2y^2 - 2xy^3]dx + [2x^3y - 3x^2y^2 - 5y^5]dy$

$[5x^4 + 3x^2y^2 - 2xy^3]\, dx + [2x^3y - 3x^2y^2 - 5y^4]\, dy$

$M = 5x^4 + 3x^2y^2 - 2xy^3$ $N = 2x^3y - 3x^2y^2 - 5y^4$

$\frac{\partial M}{\partial y} = 6x^2y - 6xy^2$ $\frac{\partial N}{\partial x} = 6x^2y - 6xy^2$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 6x^2y - 6xy^2$$

$\therefore$ The given DE is EDE

$\Rightarrow \int (5x^4 + 3x^2y^2 - 2xy^3)\, dx + \int -5y^4\, dy = c$

$\Rightarrow 5\frac{x^5}{5} + \frac{3x^3}{3}y^2 - \frac{2x^2}{2}y^3 - \frac{5y^5}{5} = c$

$\Rightarrow x^2 - y^5 - x^3y^2 - x^2y^3 = c$

5. Solve $\frac{dy}{dx} + \frac{(x+3y-4)}{(3x+9y-2)} = 0$

$\frac{dy}{dx} + \frac{x+3y-4}{(3x+9y-2)} = 0$

$(x + 3y - 4)\, dx + (3x + 9y - 2)\, dy = 0$

$M = x + 3y - 4$, $N = 3x + 9y - 2$

$\frac{\partial M}{\partial y} = 3$ $\frac{\partial N}{\partial x} = 3$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 3$$

$\therefore$ The given D.E is EDE

$\Rightarrow \int (x + 3y - 4)\, dx + \int (9y - 2)\, dy = c$

$\Rightarrow \frac{x^2}{2} + 3yx - 4x + \frac{9y^2}{2} - 2y = c$

$\Rightarrow$ Reductable exact Differential equation.

Step ①: consider the differential equation

$M(x,y)\, dx + N(x,y)\, dy = 0$

Step ②: find $\frac{\partial M}{\partial y}$ an $\frac{\partial N}{\partial x}$ and check equality

Step ③ :- If it is not equal then find $\left|\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right|$

case ① :- if $\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = f(x)$

$$I.F = e^{\int f(x)dx}$$

Multiply the Integration factor to the given DE and follow same procedure

Case ②:- if $\frac{1}{M}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = g(y)$

$$IF = e^{\int g(y)\,dy}$$

And proceed the same.

1. Solve $(4xy + 3y^2 - x)dx + x(x + 2y)dy = 0$

$(4xy + 3y^2 - x)\,dx + (x^2 + 2yx)dy = 0 \rightarrow ①$

$M = 4yx + 3y^2 - x \qquad N = x^2 + 2yx$

$\frac{\partial M}{\partial y} = 4x + 6y \qquad \frac{\partial N}{\partial x} = 2x + 2y$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\Rightarrow \frac{1}{N}\left[\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}\right] = \frac{(4x + 6y - 2x - 2y)}{x^2 + 2yx} = \frac{2x + 4y}{x^2 + 2yx} = \frac{2(x + 2y)}{x^2 + 2yx}$

$\Rightarrow \frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{2}{x} = f(x)$

$\Rightarrow IF = e^{\int f(x)\,dx} = e^{\int 2/x\, dx} = e^{2\log x} = e^{\log x^2} = x^2$

$I.F \times ① = x^2(4xy + 3y^2 - x)dx + x^2(x^2 + 2yx)dy = 0$

$(4x^3y + 3x^2y^2 - x^3)dx + (x^4 + 2yx^3)dy = 0$

$M' = 4x^3y + 3x^2y^2 - x^3 \qquad N' = x^4 + 2yx^3$

$\frac{\partial M'}{\partial y} = 4x^3 + 6x^2y \qquad \frac{\partial N'}{\partial x} = 4x^3 + 6yx^2$

$$\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x} = 4x^3 + 6x^2 y$$

The given solution is E DE

$$\Rightarrow \int (4x^3 y + 3x^2 y^2 - x^2)\, dx - \int 0\, dy = c$$

$$\Rightarrow \frac{4x^4}{4} y + \frac{3x^3}{3} y^2 - \frac{x^4}{4} = c$$

$$\Rightarrow x^4 y + x^3 y^2 - \frac{x^4}{4} = c$$

2. Solve $y(2x - y + 1) + x(3x - 4y + 3)\, dy = 0$

$$(2xy - y^2 + y)\, dx + (3x^2 - 4yx + 3x)\, dy = 0 \rightarrow ①$$

$M = 2xy - y^2 + y$ $N = 3x^2 - 4yx + 3x$

$\frac{\partial M}{\partial y} = 2x - 2y + 1$ $\frac{\partial N}{\partial x} = 6x - 4y + 3$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = \frac{6x - 4y + 3 - 2x + 2y - 1}{2xy - y^2 + y} = \frac{4x - 2y + 2}{y(2x - y + 1)}$$

$$= \frac{2(2x - y + 1)}{y(2x - y + 1)}$$

$$\Rightarrow \frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = \frac{2}{y} = g(y)$$

$$\Rightarrow I.F = e^{\int g(y)\, dy} = e^{2\int 1/y\, dy} = e^{\log y^2} = y^2$$

$\Rightarrow$ I.F × ① :- $y^2(2xy - y^2 + y)\, dx + y^2(3x^2 - 4yx + 3x)\, dy = 0$

$$(2xy^3 + y^4 + y^3)\, dx + (3x^2 y^2 - 4xy^3 + 3xy^2)\, dy = 0$$

$M' = (2xy^3 - y^4 + y^3)$ $N' = (3x^2 y^2 - 4xy^3 + 3xy^2)$

$\frac{\partial M'}{\partial y} = 6xy^2 - 4y^3 +$ $\frac{\partial N'}{\partial x} = 6x^2 y - 4y^3 + 3y^2$

$$\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x} = 6xy^2 - 4y^3 + 3y^2$$

$$\Rightarrow \int (2xy^3 - y^4 + y^3)\, dx + \int 0\, dy = c \Rightarrow x^2 y^3 - xy^4 + xy^3 = c$$

3. $(x^2+y^2+x)\,dx + xy\,dy = 0$

$(x^2+y^2+x)dx + xy\,dy = 0 \rightarrow ①$

$M = x^2+y^2+x \qquad N = xy$

$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial y} = y$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{2y-y}{xy} = \frac{y}{xy} = \frac{1}{x} = f(x)$

$\Rightarrow I.F = e^{\int f(x)dx} = e^{\int 1/x\,dx} = e^{\log x} = x$

$\Rightarrow I.F \times ① = (x^3 + xy^2 + x^2)dx + x^2y\,dy = 0$

$M^1 = x^3+xy^2+x^2 \qquad N^1 = x^2y$

$\frac{\partial M^1}{\partial y} = 2xy \qquad \frac{\partial N^1}{\partial x} = 2xy$

$$\frac{\partial M^1}{\partial y} = \frac{\partial N^1}{\partial x} = 2xy$$

$\Rightarrow \int (x^3 + y^2x + x^2)dx + \int 0\,dy = c$

$\Rightarrow \frac{x^4}{4} + \frac{y^2x^2}{2} + \frac{x^3}{3} = c$

4. Solve $y(2xy+1)\,dx - x\,dy = 0$

$(2xy^2+y)\,dx - x\,dy = 0$

$M = 2xy^2+y \qquad N = -x$

$\frac{\partial M}{\partial y} = 4xy+1 \qquad \frac{\partial N}{\partial x} = -1$

$\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = \frac{-1+4xy-1}{y(2xy+1)} = \frac{-2(x2y+1)}{y(2xy+1)} = -\frac{2}{y} = g(y)$

$\Rightarrow I.F = e^{-2\int 1/y\,dy} = e^{-2\log y} = e^{\log 1/y^2} = 1/y^2$

$\Rightarrow IF \times ① = \left(\frac{2xy^2}{y^2} + \frac{y}{y^2}\right)dx - \left(\frac{x}{y^2}\right)dy = 0$

$$(2x + 1/y)\,dx - x/y^2\,dy = 0$$

$M' = 2x + 1/y$ $\qquad$ $N' = -1/y^2$

$\frac{\partial M'}{\partial y} = \frac{-1}{y^2}$ $\qquad$ $\frac{\partial N'}{\partial x} = -1/y^2$

$\frac{\partial M'}{\partial y} = -1/y^2$ $\qquad$ $\frac{\partial N'}{\partial x} = -1/y^2$

$\Rightarrow \int (2x + 1/y)\,dx = c$

$\Rightarrow \frac{2x^2}{2} + \frac{x}{y} = c$

$\Rightarrow x^2 + \frac{x}{y} = c$

5. Solve $y(x+y)\,dx + (x+2y-1)\,dy = 0$

$(xy + y^2)\,dx + (x + 2y - 1)\,dy = 0 \rightarrow (1)$

$M = xy + y^2$ $\qquad$ $N = x + 2y - 1$

$\frac{\partial M}{\partial y} = x + 2y$ $\qquad$ $\frac{\partial N}{\partial x} = 1$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{x + 2y - 1}{x + 2y - 1} = 1 = f(x)$

$I.F = e^{\int f(x)dx} = e^{\int dx} = e^x$

$\Rightarrow I.F \times (1) \Rightarrow e^x(xy + y^2)\,dx + e^x(x + 2y - 1)\,dy = 0$

$(xye^x + e^x y^2)\,dx + (e^x x + 2ye^x - e^x)\,dy = 0$

$M' = xye^x + e^x y^2$ $\qquad$ $N' = e^x x - 2ye^x - e^x$

$\frac{\partial M'}{\partial y} = xe^x + e^x 2y$ $\qquad$ $\frac{\partial N'}{\partial x} = xe^x + 2ye^x$

$\therefore$ The given solution is EDE

$\Rightarrow \int (xye^x + e^x y^2)\,dx = c$

$\Rightarrow y(xe^x - e^x) + y^2(e^x) = c$

$\Rightarrow e^x\left[(xy - y) + y^2\right] = c$

6. Solve $(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0$

$(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0$

$M = 3x^2y^4 + 2xy$ $\qquad N = 2x^3y^3 - x^2$

$\frac{\partial M}{\partial y} = 12x^2y^3 + 2x$ $\qquad \frac{\partial N}{\partial x} = 6x^2y^3 - 2x$

$\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = \frac{6x^2y^3 - 2x - 12x^2y^3 - 2x}{(3x^2y^4 + 2xy)} = \frac{2(x^2y^3 3 + 2x)}{y(3x^2y^3 + 2x)}$

$= \frac{2}{y}$ (g)(y)

$\Rightarrow IF = e^{\int g(y)dy} = e^{\int 2/y\, dy} = e^{\log y^2} = y^2$

① × I.F = $y^2(3x^2y^4 + 2xy)\,dx + y^2(2x^3y^3 - x^2)\,dy = 0$

$(3x^2y^6 + 2xy^3)\,dx + 2x^3y^5 - x^2y^2\,dy = 0$

$M' = 3x^2y^6 + 2xy^3$ $\qquad N' = 2x^3y^5 - x^2y^2$

$\frac{\partial M'}{\partial y} = 18x^2y^5 + 6xy^2$ $\qquad \frac{\partial N'}{\partial x} = 6x^2y^5 - 2xy^2$

∴ The Reductable D.E is

$\Rightarrow \int (3x^2y^6 + 2xy^3)\,dx = c$

$\Rightarrow x^3y^6 + x^2y^3 = c$

7. Solve $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0$

$(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0 \rightarrow$ ①

$M = y^4 + 2y$ $\qquad N = xy^3 + 2y^4 - 4x$

$\frac{\partial M}{\partial y} = 4y^3 + 2$ $\qquad \frac{\partial N}{\partial x} = y^3 - 4$

$\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = \frac{(y^3 - 4 - 4y^3 - 2)}{y^4 + 2y} = \frac{3(y^3 + 2)}{y(y^4 + 2)}$ (÷ by y)

$\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = \frac{3[y^4 + 2y]}{y(y^4 + 2y)} = \frac{3}{y} = g(y)$

$\Rightarrow I.F = e^{\int g(y)dy} = e^{3\int 1/y\, dy} = e^{\log y^3} = y^3$

$I.F \times ① = y^3(y^4 + ay)dx + y^3[xy^3 + ay^4 - 4x]dy = 0$

The given Reductable D.E is

$$\Rightarrow \int (y^7 + ay^4)\,dx + \int ay^7\,dy = c$$

$$\Rightarrow xy^7 + axy^4 + \frac{ay^8}{8} = c$$

$$\Rightarrow xy^7 + axy^4 + \frac{y^8}{4} = c$$

Linear differential equation of 1st degree and 1st degree

The general Linear differential equation of 1st degree and 1st order

$$\frac{dy}{dx} + P(x)y = Q(x)$$

identify P. Q and also find $I.F = e^{\int P(x)\,dx}$

Write the Solution $y \times I.F = \int Q(x)\, IF\, dx + c$

in other form.

The Solution of $\frac{dx}{dy} + P(y)x = Q(y)$

$$\therefore IF \times x = \int Q(y)\, IF\, dy + c$$

$$I.F = e^{\int P(y)dy}$$

1. Solve $\frac{dy}{dx} - \frac{y}{x} = ax^2$

$$\frac{dy}{dx} + \left(-\frac{1}{x}\right)y = ax^2$$

$$P = -1/x \qquad Q = ax^2$$

$$I.F = \left[e^{\int P(x)dx}\right] = e^{-\int 1/x\,dx} = e^{\log(1/x)} = \frac{1}{x}$$

$$IF \times (y) = \int Q(x)\, IF\, dx + c$$

$$\Rightarrow \frac{y}{x} = \int ax^2 \frac{1}{x}\,dx + c$$

$$\Rightarrow \frac{y}{x} = \int ax^2 \frac{1}{x}\,dx + c$$

$$\Rightarrow \frac{y}{x} = \int 2x\,dx + c$$

$$\Rightarrow \frac{y}{x} = \int 2x\,dx + C$$

$$\Rightarrow \frac{y}{x} = x^2 + C$$

$$\Rightarrow y = x^2 + Cx$$

2. Solve $\frac{dy}{dx} + y \cot x = \cos x$

$$\frac{dy}{dx} + y \cot x = \cos x$$

$P = \cot x \qquad Q = \cos x$

$$\Rightarrow I.F = e^{\int P(x)\,dx} = e^{\int \cot x\,dx} = e^{\log(\sin x)} = \sin x$$

$$\Rightarrow I.F \times y = \int Q(x)\, I.F\,dx + C$$

$$\Rightarrow y \sin x = \int \cos x \sin x\,dx + c$$

$$\Rightarrow y \sin x = \frac{1}{2}\int (\sin 2x)\,dx + c$$

$$\Rightarrow y \sin x = \frac{-\cos 2x}{4} + c$$

<u>Bernoulis</u> <u>differential equation</u>

The general Solution of Bernoulis differential equation

$$\frac{dy}{dx} + P(x)\,y = Q(x)\,y^n \rightarrow (1)$$

divide eq^n (1) by y^n

$$eq^n (1) \Rightarrow \frac{1}{y^n}\frac{dy}{dx} + P(x)\frac{1}{y^{n-1}} = Q(x) \rightarrow (2)$$

Let $\frac{1}{y^{n-1}} = u$

And differentiating w r t x we get

$$\Rightarrow (-n+1)\, y^{-n}\frac{dy}{dx} = \frac{du}{dx}$$

$$\frac{1}{y^n}\frac{dy}{dx} = \frac{1}{(1-n)}\frac{du}{dx}$$

then eq^n ② $\Rightarrow \frac{1}{1-n}\frac{du}{dx} + P(x)u = Q(x)$

$$\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x)$$

$$\frac{du}{dx} + P'(x)u = Q'(x)$$

1. Solve $\frac{dy}{dx} + \frac{y}{x} = y^2 x$

$$\Rightarrow \frac{dy}{dx} + \left(\frac{y}{x}\right) = xy^2$$

$$\frac{1}{y^2}\frac{dy}{dx} + \left(\frac{1}{x}\right)\left(\frac{1}{y}\right) x \longrightarrow ①$$

let $u = 1/y \longrightarrow ②$

$D \to x$

$$\Rightarrow \frac{du}{dx} = \frac{-1}{y^2}\left(\frac{dy}{dx}\right)$$

$$\Rightarrow -\frac{du}{dx} = \frac{1}{y^2}\left(\frac{dy}{dx}\right) \longrightarrow (a)$$

Apply in eq^n (a) in eq^n ①

$$\Rightarrow -\frac{du}{dx} + \frac{u}{x} = x$$

$$\Rightarrow \frac{du}{dx} + \left(-\frac{1}{x}\right)u = -x$$

$P = -1/x$, $Q = -x$

$$\Rightarrow I.F = e^{\int P(x)dx} = e^{-\int 1/x \, dx} = e^{\log(1/x)} = 1/x$$

$$u \times IF = \int Q(x)\, I.F \, dx + c$$

$$\Rightarrow \frac{u}{x} = \int -\frac{x}{x}\, dx + c$$

$$\Rightarrow \frac{u}{x} = -\int dx + c$$

$$\Rightarrow \frac{u}{x} = -x + c$$

$$\Rightarrow u = -x^2 + cx$$

$$\Rightarrow \frac{1}{y} + x^2 = cx$$

2. Solve $\frac{dy}{dx} - y\tan x = \frac{\sin x \cos^2 x}{y^2}$

$$\Rightarrow \frac{dy}{dx} - y\tan x = \frac{\sin x \cos^2 x}{y^2}$$

$$\Rightarrow y^2 \frac{dy}{dx} + (-\tan x) y^3 = \sin x \cos^2 x \rightarrow (1)$$

Let $u = y^3$

$$\frac{du}{dx} = 3y^2 \frac{dy}{dx}$$

$$\frac{1}{3}\frac{du}{dx} = y^2 \frac{dy}{dx} \rightarrow (a)$$

apply eq^n (a) in (1)

$$\frac{1}{3}\frac{du}{dx} + (-\tan x) u = \sin x \cos^2 x$$

$$\frac{du}{dx} + (-3\tan x) u = 3\sin x \cos^2 x$$

let $P = -3\tan x \qquad Q = 3\sin x \cos^2 x$

$$I.F = e^{\int P(x)\,dx} = e^{-\int \tan x \, 3\,dx} = e^{-3\log(\sec x)} = e^{\log(1/\sec^3 x)}$$

$$= \frac{1}{\sec^3 x} = \cos^3 x$$

$$u \times I.F = \int Q(x)\, I.F\, dx + c$$

$$u \times \cos^3 x = \int 3\sin x \cos^2 x \cos^3 x\, dx + c$$

$$u \cos^3 x = 3\int (\sin x \cos^5 x)\, dx + c$$

Let $t = \cos x$

$$-dt = +\sin x\, dx$$

$$ut^3 = -3\int t^3 dt + c$$

$$ut^3 = -\frac{3t^6}{6} + c$$

$$y^3 \cos^3 x = \frac{-\cos^6 x}{2} + c$$

3. Solve $xy(1 + xy^2)\frac{dy}{dx} = 1$

$$xy(1+xy^2)\frac{dy}{dx} = 1$$

$$\frac{dx}{dy} = xy + x^2y^3$$

$$\frac{dx}{dy} - xy = x^2y^3$$

$$\frac{1}{x^2}\frac{dx}{dy} + \left(-\frac{1}{x}\right)y = y^3 \longrightarrow ①$$

Let $u = \frac{-1}{x}$

$D - y$

$$\frac{du}{dy} = \frac{1}{x^2}\frac{dx}{dy}$$

eqⁿ ① $\Rightarrow \frac{du}{dy} + uy = y^3$

$Q = y^3$

$P = y$

$$I.F = e^{\int P(y)\,dy} = e^{\int y\,dy} = e^{y^2/2}$$

$$u \times IF = \int Q(y)\, IF\, dy + c$$

$$u e^{y^2/2} = \int y^3 e^{y^2/2}\, dy + c$$

$$\frac{-e^{y^2/2}}{x} = \int (y^2 e^{y^2/2})(y\,dy) + c$$

$$\frac{-e^t}{x} = \int (2t\, e^t)\, dt + c$$

$$= 2\int (t-1)\, dt + c$$

$$\frac{-e^t}{x} = 2(t-1)\, e^t + c$$

$$\frac{-e^{y^2/2}}{x} = 2\left(y^2/2 - 1\right) e^{y^2/2} + c$$

$$\frac{-e^{y^2/2}}{x} = y^2 e^{y^2/2} - 2e^{y^2/2} + c$$

4. Solve $dy + [x \sin 2y - x^3 \cos^2 y]\,dx = 0$

$$dy + [x \sin 2y - x^3 \cos^2 y]\,dx = 0$$

$$\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$$

$$\frac{1}{\cos^2 y}\frac{dy}{dx} + \frac{x\, 2 \sin y \cos y}{\cos^2 y} = x^3$$

$$\sec^2 y \frac{dy}{dx} + \tan y (2x) = x^3 \rightarrow ①$$

Let $u = 2\tan y$

$D \rightarrow x$

$$\frac{du}{dx} = 2\sec^2 y \frac{dy}{dx}$$

$$\frac{1}{2}\frac{du}{dx} = \sec^2 y \frac{dy}{dx} \rightarrow ⓐ$$

Apply eq^n ⓐ in ①

$$\frac{1}{2}\frac{du}{dx} + ux = x^3$$

$$P = 2x \qquad Q = 2x^3$$

$$I.F = e^{\int P(x)dx} = e^{2\int x\,dx} = e^{x^2}$$

$$u \times I.F = \int Q(x)\, I.F\, dx + C$$

$$ue^{x^2} = \int e^{x^2} 2x^3\, dx + C$$

$$e^{x^2} \times 2\tan y = \int e^{x^2} x^2\, 2(x\,dx) + C$$

$$e^t \times 2\tan y = \int e^t (t)\, dt + C$$

$$= (t-1)e^t + C$$

$$2\tan y\, e^{x^2} = (x^2-1)e^{x^2} + C$$

$$2\tan y = (x^2-1) + \frac{C}{e^{x^2}}$$

5. Solve $(r\sin\theta - r^2)d\theta - \cos\theta\, dr = 0$

$$(r\sin\theta - r^2)d\theta - \cos\theta\, dr = 0$$

$$\cos\theta\, dr = (r\sin\theta - r^2)d\theta$$

$$\frac{dr}{d\theta} = \frac{r\sin\theta}{\cos\theta} - \frac{r^2}{\cos\theta}$$

$$\frac{dr}{d\theta} - r\tan\theta = \frac{-r^2}{\cos\theta}$$

$$\frac{1}{r^2}\frac{dr}{d\theta} + \left(\frac{-1}{r}\right)\tan\theta = \frac{-1}{\cos\theta} \rightarrow ①$$

let $U = -\frac{1}{r}$

$D \to \theta$

$$\frac{du}{d\theta} = \frac{1}{r^2}\frac{dr}{d\theta} \to ⓐ$$

eq^n ① $\Rightarrow \frac{du}{d\theta} + U\tan\theta = \frac{-1}{\cos\theta}$

$P = \tan\theta \qquad Q = -\frac{1}{\cos\theta}$

$I.F = e^{\int P(\theta)\,d\theta} = e^{\int \tan\theta\, d\theta} = e^{\log\sec\theta} = \sec\theta$

The solution is

$$U \times I F = \int Q(\theta)\, I.F\, d\theta + c$$

$$U\sec\theta = \int -\frac{1}{\cos\theta}\sec\theta\, d\theta + c$$

$$U\sec\theta = -\int \sec^2\theta\, d\theta + c$$

$$-\frac{\sec\theta}{r} = -\tan\theta + c$$

$$\tan\theta - \frac{\sec\theta}{r} = c$$

6. Solve $\frac{dx}{dy} = \frac{x + \sqrt{xy}}{y}$ or $\frac{dy}{dx} = \frac{y}{\sqrt{xy}+x}$

$$\frac{dx}{dy} = \frac{x}{y} + \frac{\sqrt{xy}}{y}$$

$$\frac{dx}{dy} - \frac{x}{y} = \frac{\sqrt{xy}}{y}$$

$$\frac{dx}{dy} - \frac{x}{y} = \frac{\sqrt{x}}{\sqrt{y}}$$

$$\frac{1}{\sqrt{x}}\frac{dx}{dy} + (-\sqrt{x})\frac{1}{y} = \frac{1}{\sqrt{y}} \to ①$$

$U = (-\sqrt{x})$

$D \to y$

$$\frac{du}{dy} = \frac{-1}{2\sqrt{x}}\frac{dx}{dy}$$

$$eq^n \ (1) \Rightarrow -2\frac{du}{dy} + \frac{u}{y} = \frac{1}{\sqrt{y}}$$

$$\frac{du}{dy} + \left(\frac{-1}{2y}\right)u = \frac{-1}{2\sqrt{y}}$$

$$P = \frac{-1}{2y} \qquad Q = \frac{-1}{2\sqrt{y}}$$

$$I.F = e^{\int P(x)\,dy} = e^{-\int 1/2y\,dy} = \frac{1}{\sqrt{y}}$$

$$P = \frac{-1}{2y} \qquad Q = \frac{-1}{2\sqrt{y}}$$

$$I.F = e^{\int P(y)\,dy} = e^{-\int 1/2y\,dy} = \frac{1}{\sqrt{y}}$$

$$u.I.F = \int Q.I.F\,dy + c$$

$$\frac{-\sqrt{x}}{\sqrt{y}} = \frac{1}{2}\int \frac{1}{y}\,dy + c$$

$$\frac{\sqrt{x}}{\sqrt{y}} = \frac{1}{2}\log y + c$$

$$\frac{\sqrt{x}}{\sqrt{y}} - \frac{1}{2}\log y = c$$

7. $x^3\frac{dy}{dx} - x^2y = -y^4\cos x$

$$x^3\frac{dy}{dx} - x^2y = -y^4\cos x$$

$$\frac{dy}{dx} + \left(-\frac{1}{x}\right)y = \frac{-y^4\cos x}{x^3}$$

$$\frac{1}{y^4}\frac{dy}{dx} + \left(-\frac{1}{x}\right)\frac{1}{y^3} = -\frac{\cos x}{x^3} \rightarrow (1)$$

Put $u = \frac{1}{y^3}$

$D \rightarrow x$

$$\frac{du}{dx} = \frac{-3}{y^4}\frac{dy}{dx}$$

$$-\frac{1}{3}\frac{du}{dx} = \frac{1}{y^4}\frac{dy}{dx}$$

$$-\frac{1}{3}\frac{du}{dx} - \left(\frac{u}{x}\right) = \frac{\cos x}{x^3}$$

$$\frac{dy}{dx} + \frac{3y}{x} = \frac{3\cos x}{x^3}$$

Let $P = 3/x$, $Q = \frac{3\cos x}{x^3}$

$$I.F = e^{\int P dx} = e^{\int 3/x \, dx} = e^{\log x^3} = x^3$$

The solution is given by

$$I.F \times u = \int Q \, IF \, dx + c$$

$$\frac{x^3}{y^3} = \int \frac{3\cos x}{x^3} x^3 \, dx + c$$

$$\frac{x^3}{y^3} = 3\int \cos x \, dx + c$$

$$\frac{x^3}{y^3} = 3\sin x + c$$

Orthogonal trajectory :-

Orthagonal trajectory is a curve which is $\perp^{lar}$ to the given curve

Working procedure :-

Carterian form :

Step ① :- Consider the given $F(x, y, c) = 0$

where is the parameter

Step ② :- Construct D E which is free from the paramet-er

$f(x, y, dy/dx) = 0$

Replace $\frac{dy}{dx}$ as $-\frac{dx}{dy}$ in the above we get

$g(x, y, -dx/dy) = 0$

Solve the above and we get

$G(x, y, c') = 0$

which is O. T of Given Curve

Polar form :

consider the curve $f(r, \theta, c) = \theta$

where c is parameter

consider D.E which is free from parameter

$$f(r, \theta, dr/d\theta) = 0$$

Replace $\frac{dr}{d\theta}$ as $-r^2\frac{d\theta}{dr}$ in the above we get

$$g\left(r, \theta, -r^2\frac{d\theta}{dr}\right) = 0$$

Solve the above and get

$$G(r, \theta, c') = \theta$$

which is require O.T of a given curve

1. Find orthogonal trajectory of parameter $y^2 = 4ax$

Given $y^2 = 4ax \rightarrow (1)$

$D \rightarrow x$

$$2y\frac{dy}{dx} = 4a$$

in eqn (2) $y^2 = 2y\left(\frac{dy}{dx}\right)x$

$$y = 2x\left(\frac{dy}{dx}\right) \rightarrow (2)$$

$$\frac{dy}{dx} = -\frac{dx}{dy}$$

$$y = 2x\frac{dx}{dy}$$

$$y\,dy = -2x\,dx$$

Integration OBS

$$\int y\,dy = -2\int x\,dx$$

$$\frac{y^2}{2} = -\frac{2x^2}{2} + c$$

$$y^2 = -2x^2 + 2c$$

$$y^2 + 2x^2 = 2c$$

2. Find the orthogonal trajectory of $\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1$ where λ is parameter.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2+\lambda} = 1 \rightarrow (1)$$

$D \rightarrow x$

$$\frac{2x}{a^2} + \frac{2yy_1}{b^2+\lambda} = 0$$

$$\frac{x}{a^2} + \frac{2yy_1}{b^2+\lambda} = 0$$

$$\frac{yy_1}{b^2+\lambda} = \frac{-x}{a^2}$$

$$\frac{1}{b^2+\lambda} = \frac{-x}{a^2yy_1}$$

$$b^2 - \lambda = a^2yy_1 \Rightarrow (2)$$

eqn (1) $\Rightarrow$ $$\frac{x^2}{a^2} + \frac{y^2}{\frac{-a^2yy_1}{x}} = 1$$

$$x^2 - \frac{xy}{y_1} = a^2$$

Put $y_1 = \frac{-1}{y_1}$

$$x^2 - \frac{xy}{(-1/y_1)} = a^2$$

$$x^2 + xyy_1 = a^2$$

$$xyy_1 = a^2 - x^2$$

$$yy_1 = \frac{a^2}{x} - x$$

$$y\frac{dy}{dx} = \frac{a^2}{x} - x$$

Integration OBS

$$\int y\,dy = \int \frac{a^2}{x}dx - \int x\,dx$$

$$\frac{y^2}{2} = a^2\log x - \frac{x^2}{2} + c$$

$$\int y\,dy = \int \frac{a^2}{x}dx - \int x\,dx$$

$$\frac{y^2}{2} = a^2 \log x - \frac{x^2}{2} + c$$

$$y^2 + x^2 = 2a^2 \log x + 2c$$

3. Show that $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ is a self orthogonal where λ is a parameter.

$$\frac{x^2}{a^2+\lambda^2} + \frac{y^2}{b^2+\lambda} = 1 \rightarrow ①$$

$$\frac{2x}{a^2+\lambda} + \frac{2yy_1}{b^2+\lambda} = 0$$

$$\frac{x}{a^2+\lambda} + \frac{yy_1}{b^2+\lambda} = 0$$

$$\Rightarrow x(b^2+\lambda) + (a^2+\lambda)(yy_1) = 0$$

$$\Rightarrow b^2x + \lambda x + a^2yy_1 + \lambda yy_1 = 0$$

$$\Rightarrow (x+yy_1)\lambda = -b^2x - a^2yy_1$$

$$\lambda = \frac{-b^2x - a^2yy_1}{x+yy_1}$$

$$a^2+\lambda = a^2 - \frac{b^2x - a^2yy_1}{x+yy_1} \quad ; \quad b^2+\lambda = b^2 - \frac{b^2x - a^2yy_1}{x+yy_1}$$

$$\Rightarrow a^2+\lambda = \frac{a^2(x+yy_1) - b^2x - a^2yy_1}{x+yy_1} \quad ; \quad \Rightarrow b^2+\lambda = \frac{b^2(x+yy_1) - b^2x + a^2yy_1}{x+yy_1}$$

$$\Rightarrow a^2+\lambda = \frac{a^2x + a^2yy_1 - b^2x - a^2yy_1}{x+yy_1} \quad ;$$

$$\Rightarrow b^2+\lambda = \frac{-(a^2-b^2)yy_1}{x+yy_1}$$

$$\Rightarrow a^2+\lambda = \frac{(a^2+b^2)x}{x+yy_1}$$

Then eqⁿ ① $\Rightarrow \dfrac{x^2}{\frac{(a^2-b^2)x}{x+xy_1}} + \dfrac{y^2}{\frac{-(a^2-b^2)yy_1}{x+yy_1}} = 1$

$$x(x+yy_1) - y(x+yy_1)/y_1 = a^2 - b^2$$

$$(x - yy_1)\left(x - \frac{y}{y_1}\right) = a^2 - b^2 \rightarrow ②$$

Put $y_1 = \frac{-1}{y_1}$

$$(x - y/y_1)(x + yy_1) = a^2 - b^2 \rightarrow ③$$

The eqn ② and ③ are same

∴ The eqn is self orthogonal

4. Show that $y^2 = 4a(x+a)$ is self orthogonal a is a parameter.

Consider $y^2 = 4a(x+a) \rightarrow ①$

$D \rightarrow x$

$$2y \frac{dy}{dx} = 4a$$

$$a = \frac{2yy_1}{4}$$

$$a = \frac{yy_1}{2}$$

in eqn ① $y^2 = 2yy_1(x + yy_1/2)$

$$y^2 = y(2xy_1 + yy_1^2)$$

$$y = 2xy_1 + yy_1^2 \rightarrow ②$$

This is D E of given family

Replacing $y_1 = \frac{-1}{y_1}$

$$y = 2x\left(-\frac{1}{y_1}\right) + y\left(-\frac{1}{y_1}\right)^2$$

$$y = -\frac{2x}{y_1} + \frac{y}{y_1^2}$$

$$y = \frac{-2xy_1 + y}{y_1^2}$$

$$yy_1^2 + 2xy_1 + y = 0 \rightarrow ③$$

This is the D. E of orthogonal family which is same as ② being D.E of the given family

Thus the family of parabola $y^2 = 4a(x+a)$ is orthogonal

5. find the orthogonal trajectory of $r^n = a^n \cos n\theta$ where a is a parameter.

$$r^n = a^n \cos n\theta$$

D → θ

$$n r^{n-1} \frac{dr}{d\theta} = -a^n \sin n\theta \,(n)$$

$$\frac{r^n}{r} \frac{dr}{d\theta} = -\frac{a^n \sin n\theta}{r^n}$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-a^n \sin n\theta}{a^n \cos n\theta}$$

$$\frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta$$

Replace $\frac{dr}{d\theta} = -r^2 \frac{d\theta}{dr}$

$$\frac{1}{r}\left[r^2 \frac{d\theta}{dr}\right] = \tan n\theta$$

$$-\frac{1}{r} dr = \frac{1}{\tan n\theta} d\theta$$

Integration OBS

$$\int \frac{1}{r} dr = \int \frac{1}{\tan n\theta} d\theta$$

$$\log r = \int \cot n\theta \, d\theta$$

$$\log r = \int \cot n\theta \, d\theta$$

$$\log r = \frac{\log(\sin n\theta)}{n} + \log k$$

$$n \log r^n = \log \sin n\theta + n \log k$$

$$\log r^n = \log(\sin n\theta) + \log k^n$$

$$\log r^n = \log[\sin n\theta \; k^n]$$

$$r^n = k^n \sin n\theta$$

6. Find the orthogonal trajectory of $r^n = a^n \sin n\theta$ where a is parameter.

$$r^n = a^n \sin n\theta$$

$$D \rightarrow \theta$$

$$\frac{dr}{d\theta} n r^{n-1} = a^n (\cos n\theta) n$$

$$\frac{r^n}{r} \frac{dr}{d\theta} = a^n \cos n\theta$$

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{a^n \cos n\theta}{r^n}$$

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{a^n \cos n\theta}{a^n \sin n\theta}$$

$$\frac{1}{r} \frac{dr}{d\theta} = \cot n\theta$$

Replace $\left(\frac{dr}{d\theta}\right) = -r^2 \frac{d\theta}{dr}$

$$\frac{1}{r}\left(-r^2 \frac{d\theta}{dr}\right) = \cot n\theta$$

$$\frac{1}{r}(dr) = \frac{1}{\cot n\theta} d\theta$$

Integration on B S

$$-\int \frac{1}{r} dr = -\int \tan n\theta \, d\theta$$

$$\log k - \log r = \frac{+\log(\sec \theta n)}{n}$$

$$n \log k - n \log r = \log \sec n\theta$$

$$\log k^n - \log r^n = \log \sec n\theta$$

$$\log(r^n) + \log k^n = \log(\sec n\theta)$$

$$\log\left(\frac{k^n}{r^n}\right) = \log(\sec n\theta)$$

$$r^n = k^n \frac{1}{\sec n\theta}$$

$$\Rightarrow r^n = k^n \cos n\theta$$

7. Find the orthogonal trajectory of $r^n \sin n\theta = a^n$ where a is parameter.

$$r^n \sin n\theta = a^n$$

$D \to \theta$

$$n r^{n-1} \frac{dr}{d\theta} \sin n\theta + r^n \cos n\theta (n) = 0$$

$$\frac{r^n}{r} \frac{dr}{d\theta} \sin n\theta = -r^n \cos n\theta$$

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{-\cos n\theta}{\sin n\theta}$$

$$\frac{1}{r} \frac{dr}{d\theta} = -\cot n\theta$$

Replace $\frac{dr}{d\theta} = -r^2 \frac{d\theta}{dr}$

$$\frac{1}{r} \cdot \frac{-r^2 d\theta}{dr} = -\cot n\theta$$

$$r \frac{d\theta}{dr} = \cot n\theta$$

$$\frac{1}{r} dr = \frac{1}{\cot n\theta} d\theta$$

Integration O B S

$$\int \frac{1}{r} dr = \int \tan n\theta \, d\theta$$

$$\log r = \frac{\log (\sin n\theta)}{n} + \log k$$

$$n \log r = -\log (\cos n\theta) + n \log k$$

$$\log r^n + \log (\cos n\theta) = \log k^n$$

$$\log (r^n \cos n\theta) = \log k^n$$

$$r^n (\cos n\theta) = k^n$$

8. find the orthogonal trajectory of family of curve $r = 2a \cos\theta$ where 'a' is parameter.

$$r = 2a\cos\theta \rightarrow (1)$$

$D \rightarrow \theta$

$$\frac{dr}{d\theta} = -2a\sin\theta \rightarrow (2)$$

divide eq (2) and (1)

$$r\frac{d\theta}{dr} = -\cot n\theta$$

$$\frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta$$

Replace $\left(\frac{dr}{d\theta}\right) = -r^2\frac{d\theta}{dr}$

$$-\frac{1}{r}\, r^2\frac{d\theta}{dr} = -\tan n\theta$$

$$r\frac{d\theta}{dr} = \tan n\theta$$

$$\frac{1}{r}\,dr = \frac{1}{\tan n\theta}\,d\theta$$

Integrate OBS

$$\int\frac{1}{r}\,dr = \int\frac{1}{\tan n\theta}\,d\theta$$

$$\log r = \int\cot n\theta\,d\theta$$

$$\log r = \log\sin n\theta + \log k$$

$$\log r = \log k\sin n\theta$$

$$r = k\sin\theta$$

9. Find the orthogonal trajectory of curve $r = a(1-\cos\theta)$ where 'a' is parameter.

$$r = a(1-\cos\theta) \rightarrow (1)$$

$D \rightarrow \theta$

$$\frac{dr}{d\theta} = a\sin\theta \rightarrow (2)$$

eqn (2) by (1)

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta}$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2}$$

$$\frac{1}{r}\frac{dr}{d\theta} = \tan\theta/2$$

Replace $\frac{dr}{d\theta} = -r^2\frac{d\theta}{dr}$

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \tan\theta/2$$

$$r\frac{d\theta}{dr} = -\tan\theta/2$$

$$\frac{1}{r}dr = -\frac{1}{\tan(\theta/2)}d\theta$$

Integrate oBS

$$\int\frac{1}{r}dr = -\int\cot(\theta/2)\,d\theta$$

$$\log r = -\frac{\log(\sec(\theta/2))}{(1/2)} + \log k$$

$$\log r = -2\log\sec(\theta/2) + \log k$$

$$\log r = \log\cos^2(\theta/2) + \log k$$

$$\log r = \log(k\cos^2\theta/2)$$

$$r = k\cos^2\theta/2$$

$$r = k\left(\frac{1+\cos\theta}{2}\right)$$

$$r = \frac{k}{2}[1+\cos\theta]$$

$$r = b[1+\cos\theta]$$

10. Find the orthogonal trajectory of $r^n\cos n\theta = a^n$ where 'a' is parameter.

$$r^n\cos n\theta = a^n$$

$$D \to \theta$$

$$nr^{n-1}\frac{dr}{d\theta}(\cos n\theta - r^n\sin n\theta(n)) = 0$$

$$n\frac{r^n}{r}\frac{dr}{d\theta}\cos n\theta = r^n\sin n\theta\,(n)$$

$$\frac{1}{r}\frac{dr}{d\theta}\cos n\theta = \sin n\theta$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin n\theta}{\cos n\theta}$$

$$\frac{1}{r}\frac{dr}{d\theta} = \tan n\theta$$

Replace $\frac{dr}{d\theta} = -\frac{d\theta}{dr} r^2$

$$\frac{1}{r}\left[-r^2 \frac{d\theta}{dr}\right] = \tan n\theta$$

$$\frac{1}{\tan n\theta} d\theta = -\frac{1}{r} dr$$

Integration D BS

$$\int \cot n\theta \, d\theta = -\int \frac{1}{r} dr$$

$$\log(\sin n\theta)\frac{1}{n} = -\log r + \log k$$

$$\log(\sin n\theta) + n\log r = n\log k$$

$$\log(\sin n\theta \, r^n) = \log k^n$$

$$r^n \sin n\theta = k^n$$

Newton's law of cooling:-

Let 'T' be the temperature of a body, T_0 be the temp. of surrounding medium at any time 't', by the property of the Newton's law of cooling the rate of temperture of body is directly porpostional to difference between temperature of the and it's surrounding medium.

$$\frac{dT}{dt} \propto (T - T_0)$$

$$\frac{dT}{dt} = -k[T - T_0]$$

$$dT = -k[T - T_0]dt$$

Integration D Bs

$$\int dT = -\int k[T - T_0]dt$$

$$\int \frac{1}{[T - T_0]} dT = -k\int dt$$

$$\log[T - T_0] = -kt + c$$

$$T - T_0 = e^{-Kt + c}$$

$$T = T_0 + e^{-Kt + c}$$

$$T = T_0 + e^{-Kt} e^{c}$$

$$\boxed{T = T_0 + \lambda e^{-Kt}}$$

1. a copper ball originally at 180° cool down 80°c in 20 min. 40°c what will be the ball after 40 min from the origin.

$T_A = 40°c$

$t = 0$, $T = 80°c$

$t = 20$ $T = 60°c$

Suppose T be a temperture of copper ball and T_A is the temperture of the air at any time (t)

w.k.T

$$T = T_0 + \lambda e^{-Kt} \rightarrow (1)$$

given temp/. of air $T_0 = 40°c$

$$eq^n (1) \Rightarrow T = 40 + \lambda e^{-Kt} \rightarrow (2)$$

Given $T = 80°c$ when $t = 0$

$$eq^n (2) \Rightarrow 80 = 40 + \lambda e^{-K(0)}$$

$$\Rightarrow \lambda = 40$$

$$eq (2) \Rightarrow T = 40 + 40e^{-Kt} \rightarrow (3)$$

$T = 60°c$ $t = 20°$

$$(3) \Rightarrow 40 + 40e^{-Kt} = 60$$

$$20 = 40e^{-20K}$$

$$e^{-K20} = 0.5$$

$$-20K = \log_e(0.5)$$

$$K = 0.03465$$

③ ⇒ $T = 40 + 40 e^{(-0.0346)t}$ → ④

when $t = 40$ min

eq ④ ⇒ $T = 40 + 40 e^{(-0.0346)40}$

$T = 40 + 40 e^{-1.384}$

$T = 50°c$

2. The temperature of air is 30°C a metal ball cools from 100°c to 70°c in 15 min find How many long will it taken for metal to reach the temperature of 40°c

⇒ Suppose T be the temperature of metal ball T_0 is temp/. of air at time 't'

w.k.t $T = T_0 + \lambda e^{-Kt}$ → ①

given that $T_0 = 30°c$

$T = 30 + \lambda e^{-Kt}$ → ②

Given $T = 100°c$ at $t = 0$

② ⇒ $100 = 30 + \lambda e^{-Kt}$

$\lambda = 70$

② ⇒ $T = 30 + 70 e^{-Kt}$ → ③

$T = 70°c$, $t = 15$ min

③ ⇒ $40 = 30 + 70 e^{-(0.0373t)}$

$\frac{10}{70} = e^{-0.0373t}$

$-0.0373t = \ln(0.1428)$

$-0.0373t = -1.9463$

$t = 52$ mic 18 ec

3. A body is of heated 110°c and placed in air at 10°c after 1 hour, it's temp. becomes 60°c how much additional time is required to cool.

⇒ Suppose T be the temp/. of the body, T_0 is temp/. of air

$T = T_0 + \lambda e^{-Kt} \rightarrow (1)$

$T_0 = 10^\circ$

$(1) \Rightarrow T = 10 + \lambda e^{-Kt} \rightarrow (2)$

$T = 110^\circ C$ at $t = 0$

$(2) \Rightarrow 110 = 10 + \lambda e^{-0(K)}$

$\lambda = 100$

$T = 10 + 110e^{-60K} \rightarrow (3)$

Given, $T = 60^\circ C$, $t = 60^\circ$

$(3) \Rightarrow 60 = 10 + 110e^{-60K}$

$0.5 = e^{-60K}$

$-60K = -0.69314$

$K = 0.01155$

$(3) \Rightarrow T = 10 + 100e^{-(0.01155t)} \rightarrow (4)$

also given $T = 30$ $t = ?$

$30 = 10 + 100e^{-(0.01155)t}$

$\ln(0.2) = -0.01155t$

$t = 139.345$

t = 2 hours 19 min 34 sec

4. A bottle of mineral water at a room temp. 72F is kept in refrigitor where the temp. is 44F after half of an hour, what are cooled 61F, what is the temp. of mineral water another half an hour.

$\Rightarrow$ From newton's law

$T = T_0 + \lambda e^{-Kt} \rightarrow (1)$

$T_0 = 44F$

$(1) \Rightarrow T = 44 + \lambda e^{-Kt} \rightarrow (2)$

$T = 72F$, when $t = 0$

$72 = 44 + \lambda e^{-0K}$

$\lambda = 28$

② $\Rightarrow T = 44 + 28e^{-Kt} \rightarrow$ ③

$T = 61F$, $t = 30$

③ $\Rightarrow 61 = 44 + 28e^{-30K}$

$K = 0.01663$

③ $\Rightarrow T = 44 + 28e^{-0.01663(60)}$

$T = 44 + 10.323$

$T = 54.32°F$

5. A body in at 25°C whose from 100°C to 75°C in one minute. find the temperture of a body at 3 minutes.

$\Rightarrow$ From Newton's law

$T = T_0 + \lambda e^{-Kt} \rightarrow$ ①

Given that air temperture $T_0 = 25$

① $\Rightarrow T = 25 + \lambda e^{-Kt} \rightarrow$ ②

They given that $T = 100°C$ at $t = 0$

② $\Rightarrow 100 = 25 + \lambda e^{-K(0)}$

$\lambda = 75$

② $\Rightarrow T = 25 + 75e^{-Kt} \rightarrow$ ③

$T = 75F$ at $t = 1$

③ $\Rightarrow 75e^{-K} + 25 = 75$

$e^{-K} = 0.666$

$K = 0.406$

③ $\Rightarrow T = 25 + 75e^{-0.406t} \rightarrow$ ④

given $T = ?$ $t = 3$

$T = 25 + 75e^{(-0.406 \times 3)}$

$T = 25 + 22.186$

$T = 47.186C°$

Flow of electricity :- [L-R circuits]

The electrical circuit may have 3 passive elements they are resistance (R), inductance (L), capacitance (C) and active element be voltage source with a emf source with (E) to a current (I) at any time 't'.

①. A Series circuit with resistance (R), inductance (L) and emf (E) governed by D.E. $L\frac{di}{dt} + Ri = E$, where L, R constant and intial, the current is zero. find the current at any time (t).

⇒ Given that resistance b/w current, resistance, and inductance, E is $L\frac{di}{dt} + Ri = E$, R, L constant

$$L\frac{di}{dt} + Ri = E \rightarrow ①$$

$$\frac{di}{dt} + \left(\frac{R}{L}\right)i = \frac{E}{L} \rightarrow ②$$

$$P = R/L \qquad Q = E/L$$

$$I.F = e^{\int P dt} = e^{\int R/L \, dt} = e^{(R/L)t}$$

The Solution is given by $i \times I.F = \int Q \, IF \, dt + c$

$$i \times e^{(R/L)t} = \int \frac{E}{L} e^{(R/L)t} dt + c$$

$$i \times e^{(R/L)t} = \frac{E}{L} \frac{e^{(R/L)t}}{R/L} + K$$

$$i \times e^{(R/L)t} = E \frac{e^{(R/L)t}}{R} + K \rightarrow ③$$

$$i(t) = \frac{E}{R} + Ke^{-R/L\,t} \rightarrow ④$$

Given $i = 0$, $t = 0$

$$0 = \frac{E}{R} + Ke^{-0}$$

$$K = E/R \rightarrow ⓐ$$

Substitute eq^n (a) in (4)

$$i(t) = \frac{E}{R} - \frac{E}{R} e^{-(R/L)t}$$

$$i(t) = \frac{E}{R}\left[1 - e^{-(R/L)t}\right]$$

2. An Inductance 2H and resistance 20Ω are Connected to a Series emf 'E' holds if the current is intial zero, when t = 0 find the current at end 0.01 sec if E = 100 volts ?

W.K.T the D.E of a L-R Circuit is

$$L\frac{di}{dt} + Ri = E \rightarrow (1)$$

Gives the Current at any time 't' is

$$i(t) = \frac{E}{R}\left[1 - e^{-(R/L)t}\right] \longrightarrow (2)$$

Given E = 100V, L = 2H, R = 20Ω to find Current t = 0.01 sec

$$i_{(0.01)} = \frac{100}{20}\left[1 - e^{-(20/2)(0.01)}\right]$$

$$= 5\left[1 - e^{-0.1}\right]$$

$$= 5\left[1 - 0.9048\right]$$

$$i_{(0.01)} = 5\left[0.09516\right]$$

$$i_{(0.01)} = 0.4758 \text{ Amp}$$

3. L-R Series Circuit D.E acted on by an emf 'E' Sinωt, Satisfy if there is no current in the circuit intial, after the value of current any time (t).

⇒ The given D.E of L-R Circuit

$$L\frac{di}{dt} + Ri = E \sin\omega t$$

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}\sin\omega t$$

$$P = R/L \quad , \quad Q = E/L \sin\omega t$$

$$I.F = e^{\int P dt} = e^{\int R/L\, dt} = e^{(R/L)t}$$

$$i \times IF = \int Q\, I.F\, dt + K$$

$$i e^{(R/L)t} = \int \frac{E}{L} \sin\omega t \; e^{(R/L)t} \, dt + K$$

$$i e^{(R/L)t} = \frac{E}{L} \int e^{(R/L)t} \sin\omega t \, dt + K$$

$$i e^{(R/L)t} = \frac{E}{L} \frac{e^{(R/L)t}}{\sqrt{\frac{R^2}{L^2} + \omega^2}} \sin[\omega t - \tan^{-1}(\omega/R_L)] + K$$

$$i e^{(R/L)t} = \frac{E}{L} \frac{e^{(R/L)t}}{\sqrt{R^2 + \omega^2 + L^2}} \sin[\omega t - \tan^{-1}(\omega L/R)] + K$$

$$i = \frac{E}{\sqrt{R^2 + \omega^2 L^2}} \sin[\omega t - \tan^{-1}(\omega L/R)] + K e^{(R/L)t} \rightarrow (1)$$

w.k.T , $t = 0$, $i = 0$

$$0 = \frac{E}{\sqrt{R^2 + \omega^2 L^2}} \sin(\omega(0) - \tan^{-1}(\omega L/R)] + K e^{(R/L)0}$$

$$\frac{E}{\sqrt{R^2 + \omega^2 L^2}} \sin(-\phi) + K = 0 \qquad \therefore -\phi = -\tan^{-1}(\omega L/R)$$

$$K = \frac{E}{\sqrt{R^2 + \omega^2 L^2}} (\sin\phi) \rightarrow (a)$$

Substitute eq^n (a) in (1)

$$i = \frac{E}{\sqrt{R^2 + \omega^2 L^2}} \sin[\omega t - \phi] + \frac{E}{\sqrt{R^2 + \omega^2 L^2}} \sin\phi \; e^{(R/L)t}$$

4. Solve the D.E $L \frac{di}{dt} + Ri = 200(\sin(300t))$ when $L = 0.05$ and $R = 100\,\Omega$. find the value of current (I) at any time 't', intial no current in circuit what values does apporach after long time.

$\Rightarrow$ w.k.T

for D.E $L\frac{di}{dt} + Ri = 200 \sin(300)t$

The value of current at any time (t) in initial current is zero.

$$i(t) = \frac{E}{\sqrt{R^2+\omega^2L^2}} \sin(\omega t - \phi) + e^{(R/L)t} \sin\phi \rightarrow ①$$

by comparing given eq^n with general

$R = 100$, $L = 0.05$, $E = 200$, $\omega = 300$

$$\sqrt{R^2+\omega^2L^2} = \sqrt{(100)^2 + (300^2)(0.05)^2} = 101.1187$$

$$\phi = \tan^{-1}(\omega L/R) = \tan^{-1}\left(\frac{300 \times 0.05}{100}\right) = 8.5307$$

$$① \Rightarrow i(t) = \frac{200}{101.1187} \sin(300t - 8.5307) + e^{(100/0.05)t} \sin(8.5307)$$

$$\Rightarrow i(t) = 1.9778 \sin(300t - 8.5307) + e^{2000t}(0.14833)$$

Non Linear differential equation :-

For $P = \frac{dy}{dx}$ the polynomial

$$A_n p^n + A_1 p^{n-1} + A_2 p^{n-2} + A_3 p^{n-3} + \dots + A_n p^{n-n} = 0 \rightarrow ①$$

is called Linear D.E

For which $A_0, A_1, A_2, A_3 \dots A_n$ are the f^n of x and y then the solution of eq^n can be done by the method of Solvable P. as fallow.

$$① \Rightarrow [P - f_1(x,y)][P - f_2(x,y)], [P - f_3(x,y)] \dots [P - f_n(x,y)] = 0$$

$$\Rightarrow P - f_1(x,y) = 0 \dots P - f_2(x,y) = 0 \dots\dots [P - f_n(x,y)] = 0$$

$$\Rightarrow P = f_1(x,y) \dots P = f_2(x,y) \dots P = f_n(x,y)$$

$$\Rightarrow F_1(x,y,c) = 0, \dots F_2(x,y,c_2) = 0 \dots F_n(x,y,c_n) = 0$$

$\therefore F_1(x, y, c_1), F_2(x, y, c_2) \ldots \ldots F_n(x, y, c_n) = 0$

1. $\left(\frac{dy}{dx}\right)^2 - 7\left(\frac{dy}{dx}\right) + 12 = 0$

$$\left(\frac{dy}{dx}\right)^2 - 7\left(\frac{dy}{dx}\right) + 12 = 0$$

Let $P = \frac{dy}{dx}$

$$P^2 - 7P + 12 = 0$$

$$P(P-4) - 3(P-4) = 0$$

$$P-4 = 0 \quad , \quad P-3 = 0$$

$$P = 4 \quad , \quad P = 3$$

$$\frac{dy}{dx} = 4 \quad , \quad \frac{dy}{dx} = 3$$

$$dy = 4dx \qquad dy = 3dx$$

Integrate OBS

$$\int dy = \int 4dx \qquad \int dy = \int 3dx$$

$$y = 4x + c_1 \qquad y = 3x + c_2$$

$$y = 4x - c_1 = 0 \qquad y = 3x - c_2 = 0$$

$\therefore$ The Solution is

$$(y - 4x - c_1)(y - 3x - c_2) = 0$$

2. Solve $y\left(\frac{dy}{dx}\right)^2 + (x-y)\left(\frac{dy}{dx}\right) - x = 0$

$$y\left(\frac{dy}{dx}\right)^2 + (x-y)\left(\frac{dy}{dx}\right) - x = 0 \rightarrow (1)$$

Let $P = \frac{dy}{dx}$

in eqn (1) $yp^2 + (x-y)P - x = 0$

$$yp^2 + xp - yp - x = 0$$

$$yp^2 - yp + xp - x = 0$$

$$yp(p-1) + x(p-1) = 0$$

Case ① :-

$p - 1 = 0$

$\Rightarrow p = 1$

$\Rightarrow \frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = 1$

$dy = dx$

$\Rightarrow \int dy = \int dx$

$\Rightarrow y - x + c_1 = 0$

Case ② :-

$yp + x = 0$

$\Rightarrow yp = -x$

$\Rightarrow y\frac{dy}{dx} = -x$

$\Rightarrow \int y\,dy = -\int x\,dx$

$\Rightarrow \frac{y^2}{2} = \frac{-x^2}{2} + c_2$

$\Rightarrow \frac{1}{2}[y^2 + x^2] = c_2$

$y^2 + x^2 - 2c_2 = 0$

$\therefore$ The Solution is given by

$$(y - x - c_1)(y^2 + x^2 - 2c_2) = 0$$

3. Solve $\quad xy\left(\frac{dy}{dx}\right)^2 - (x^2+y^2)\frac{dy}{dx} + xy = 0$

$$xy\left(\frac{dy}{dx}\right)^2 - (x^2+y^2)\left(\frac{dy}{dx}\right) + xy = 0 \rightarrow ①$$

Let $p = \frac{dy}{dx}$

eqⁿ ① $\Rightarrow xy(p^2) - (x^2+y^2)(p) + xy = 0$

$$xp[yp - x] - y[yp - x] = 0$$

$$(xp - y)(yp - x) = 0$$

Case ① :-

$px - y = 0$

$px = y$

$x\frac{dx}{dy} = y$

$\int \frac{1}{x}dx = \int \frac{1}{y}dy$

$\log x = \log y + \log c_1$

$\log x = \log(y_1 c_1)$

$x = yc_1$

Case ② :-

$yp - x = 0$

$yp = x$

$y\frac{dy}{dx} = x$

$\int y\,dy = \int x\,dx$

$\frac{y^2}{2} = \frac{x^2}{2} + c_2$

$y^2 - x^2 - 2c_2 = 0$

$\therefore$ The Solution is $(x - yc_1)(y^2 - x^2 - 2c_2) = 0$

4. Solve $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$

$$\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x} \rightarrow (1)$$

Let $P = \frac{dy}{dx}$ $\quad \frac{dx}{dy} = \frac{1}{P}$

$$(1) \Rightarrow P - \frac{1}{P} = \frac{x}{y} - \frac{y}{x}$$

$$\frac{P^2-1}{P} = \frac{x^2-y^2}{xy}$$

$$(P^2-1)xy = P(x^2-y^2)$$

$$yxp^2 - xy = px^2 - py^2$$

$$xy[p^2-1] = p[x^2-y^2] = 0$$

$$xyp^2 - xy - x^2p + y^2p = 0$$

$$xy[yp-x] + xp[yp-x] = 0$$

$$(xp+y)(yp-x) = 0$$

$xp + y = 0$

$xp = -y$

$p = -\frac{y}{x}$

$\frac{dy}{dx} = -\frac{y}{x}$

$\int \frac{1}{y} dy = -\int \frac{1}{x} dx$

$\log y + \log x = \log c$

$\log\left(\frac{y}{x}\right) = \log c_1$

$\frac{y}{x} = c_1$

$y = xc_1$

$y - c_1 x = 0$

$yp - x = 0$

$yp = x$

$p = \frac{x}{y}$

$\frac{dy}{dx} = \frac{x}{y}$

$\int y\,dy = \int x\,dx$

$\frac{y^2}{2} = \frac{x^2}{2} + c_2$

$y^2 - x^2 = 2c_2$

$y^2 - x^2 - 2c_2 = 0$

∴ The solution is $(y - c_1 x)(y^2 - x^2 - 2c_2) = 0$

5. Solve $p^2 + 2py\cot x = y^2$

$$p^2 + 2py\cot x - y^2 = 0 \rightarrow (1)$$

$$p = \frac{-2y\cot x \pm \sqrt{4y^2\cot^2 x - 4(-y^2)}}{2(1)}$$

$$p = \frac{-2y\cot x \pm 2y\sqrt{\cot^2 x + 1}}{2}$$

$$p = -y\cot x \pm y\,\mathrm{cosec}\,x$$

Case (1) :- $p = -y\cot x + y\,\mathrm{cosec}\,x$

$$\frac{dy}{dx} = -y[\mathrm{cosec}\,x - \cot x]dx$$

$$\int \frac{1}{y}dy = \int \mathrm{cosec}\,x - \cot x\,dx$$

$$\log y = \log\tan(x/2) - \log\sin x + \log c_1$$

$$= \log\left[c_1 \frac{\tan(x/2)}{\sin x}\right]$$

$$= \log\left[c_1 \frac{\sin(x/2)/\cos(x/2)}{2\sin(x/2)\cos(x/2)}\right]$$

$$= \log\left[\frac{c_1}{2\cos^2(x/2)}\right]$$

$$= \log\left[\frac{c_1}{1+\cos x}\right]$$

$$y = \frac{c_1}{(1+\cos x)}$$

$$y(1+\cos x) - c_1 = 0$$

Case (2) :- $p = -y\cot x - y$

$$p = -y\cot x - y\,\mathrm{cosec}\,x$$

$$p = -y[\cot x + \mathrm{cosec}\,x]$$

$$\frac{dy}{dx} = -y[\cot x + \mathrm{cosec}\,x]$$

$$-\int \frac{1}{y}dy = \int(\cot x + \mathrm{cosec}\,x)dx$$

$$-\int \frac{1}{y}\,dy = \int (\cot x + \operatorname{cosec} x)\,dx$$

$$-\log y = \log(\sin x) + \log\tan(x/2) + \log C_2$$

$$\log(1/y) = \log[\sin x \cdot \tan x/2 \; C_2]$$

$$\Rightarrow y(1-\cos x) - C_2 = 0$$

$\therefore$ The Solution is $(y(1+\cos x)-C_1)(y(1-\cos x)-C_2=0$

$\Rightarrow$ Clairaut's equation :-

The D.E of the form $y = P(x) + f(x)$ is said to be Clairaut's form and it's Solution is obtained by replacing $P \rightarrow C$ the Solution becomes, $y = f(x) + f(x)$ is called general solution of Clairaut's and D.E differentiate partially w r t c, we get function interms of c and again replace or substitute in given solution it is called singular Solution.

① . Solve the clairaut's equation $y = px + \frac{a}{p}$

given $y = Px + \frac{a}{p} \rightarrow$ ①

which is in clairaut's form $y = P(x) + F(x)$

$\therefore$ The Solution is $y = Cx + \frac{a}{c} \rightarrow$ ②

$$D \rightarrow c$$

$$x - \frac{a}{c^2} = 0$$

$$x = \frac{a}{c^2}$$

$$c^2 = \frac{a}{x}$$

$$c = \sqrt{\frac{a}{x}}$$

$\therefore$ The Sigular solution is

$$② \Rightarrow y = x\sqrt{\frac{a}{x}} + \frac{a}{\frac{\sqrt{a}}{\sqrt{x}}}$$

$\Rightarrow y = \sqrt{x}\sqrt{a} + \sqrt{x}\sqrt{a} \Rightarrow y = 2\sqrt{ax} \Rightarrow y^2 = 4ax$ //

2. Show that equation $xp^2 + px - py - y + 1 = 0$ where use Clairaut's equation.

$$xp^2 + px - py + 1 - y = 0$$

$$xp(p+1) - y(p+1) = -1$$

$$(p+1)(xp - y) = -1$$

$$px - y = \frac{-1}{(p+1)}$$

$$-y = -px - \frac{1}{p+1}$$

$$y = px + \frac{1}{p+1} \rightarrow (1)$$

eqn (1) is Clairaut's equation

$$y = cx + \frac{1}{c+1}$$

$$D \rightarrow x$$

$$0 = x - \frac{1}{(c+1)^2}$$

$$\frac{1}{(c+1)^2} = x$$

$$c+1 = \frac{1}{\sqrt{x}}$$

∴ The solution of the following

$$y = \left(\frac{1}{\sqrt{x}} - 1\right)x + \frac{1}{\frac{1}{\sqrt{x}} - 1 + 1}$$

$$y = \sqrt{x} - x + \sqrt{x}$$

$$y = 2\sqrt{x} - x$$

3. Show that the equation $xp^3 - yp^2 + 1 = 0$

$$xp^3 - yp^2 + 1 = 0$$

$$p^2[xp - y] = -1$$

$$px - y = \frac{-1}{p^2}$$

$$-y = \frac{-1}{p^2} + px \rightarrow (1)$$

(1) is a clairaut's equation

$$y = cx + \frac{1}{c^2}$$

Partial D → c

$$0 = x - \frac{2}{c^3}$$

$$\frac{2}{c^3} = x$$

$$c^3 = \frac{2}{x}$$

$$c = \sqrt[3]{\frac{2}{x}}$$

The Solution is

$$y = \sqrt[3]{2/x} + (x/2)^{2/3}$$

4. $(px-y)(py+x) = 2p$ is reduced to clairaut's form taking $X = x^2$ $Y = y^2$

⇒ Given that $(px-y)(py+x) = 2p \rightarrow (1)$

$$X = x^2 \quad Y = y^2$$

$$\frac{dX}{dx} = 2x \quad \frac{dY}{dy} = 2y$$

W.K.T

$$p = \frac{dy}{dx}$$

$$p = \frac{dy}{dY} \frac{dY}{dX} \frac{dX}{dx}$$

$$p = \frac{1}{2y} P 2x$$

$$p = \frac{x}{y} P$$

$$p = \frac{\sqrt{X}}{\sqrt{Y}} P$$

$$(1) \Rightarrow \left[\frac{\sqrt{X}}{\sqrt{Y}} P\sqrt{X} - \sqrt{Y}\right]\left[\frac{\sqrt{X}}{\sqrt{Y}} P\sqrt{Y} + \sqrt{X}\right] = 2\frac{\sqrt{X}}{\sqrt{Y}} P$$

$$\left[\frac{px-y}{\sqrt{y}}\right]\left[(p+1)\right]\sqrt{x} = \frac{2\sqrt{x}p}{\sqrt{y}}$$

$$(px-y)(p+1) = 2p$$

$$px-y = \frac{2p}{p+1}$$

$$y = px + \left(\frac{-2p}{p+1}\right)$$

The reduced eq^n is

$$y = cx + \left(\frac{-2c}{c+1}\right)$$

5. Solve $e^{4x}(p-1) + e^{2y}p^2 = 0$ by using Subsitimtution $u = 2e^{2x}$ and $v = e^{2y}$

$$\Rightarrow e^{4x}(p-1) + e^{2y}p^2 = 0$$

$$u = e^{2x}, \quad v = e^{2y}$$

$$\frac{du}{dx} = 2e^{2x} \qquad \frac{dv}{dy} = 2e^{2y}$$

w.k.T

$$p = \frac{dy}{dx}$$

$$p = \frac{dy}{dy}\frac{dv}{du}\frac{du}{dx}$$

$$p = \frac{1}{2e^{2y}} \quad P \; 2e^{2x}$$

$$p = \frac{e^{2x}}{e^{2y}} P$$

$$p = \frac{u}{v} P$$

$$① \Rightarrow e^{4x}\left(\frac{u}{v}P - 1\right) + e^{2y}\left(\frac{u^2P^2}{v^2}\right) = 0$$

$$u^2\left(\frac{Pv - v}{v}\right) + v\left(\frac{u^2}{v^2}\right)P^2 = 0$$

$$Pu - v = -P^2$$

$$v = P^2 + Pu$$

The resultant equation is

$$V = UC + C^2$$

$$e^{2y} = Ce^{2x} + c^2$$

6. Find the general and Singular Solution $x^2[y-Px] = p^2y$ by taking into clairaut's form using substitution $X = x^2$, $Y = y^2$

$$\Rightarrow x^2[y-Px] = p^2 y \rightarrow ①$$

$$X = x^2 \qquad Y = y^2$$

$$\frac{dX}{dx} = 2x \qquad \frac{dY}{dy} = 2y$$

$$P = \frac{dY}{dX}$$

$$p = \frac{dy}{dY} \frac{dY}{dX} \frac{dX}{dx}$$

$$p = \frac{1}{2y} P\, 2x$$

$$p = \frac{\sqrt{X}}{\sqrt{Y}} P$$

$$① \Rightarrow X\left[\sqrt{Y} - \frac{\sqrt{X}}{\sqrt{Y}} P\sqrt{X}\right] = \frac{X}{Y} P^2\sqrt{Y}$$

$$X\left[\frac{Y - XP}{\sqrt{Y}}\right] = \frac{XP^2}{\sqrt{Y}}$$

$$Y - PX = P^2$$

∴ The Solution is in Clairat's Solution

$$y^2 = cx^2 + c^2 \qquad Y = cX + c^2 \rightarrow ②$$

$$D \rightarrow c$$

$$0 = x^2 + 2c$$

$$2c = x^2$$

$$c = x^2/2$$

∴ The Singular Solution is

$$y^2 = \left(-\frac{x^2}{2}\right) x^2 + \left(\frac{x^2}{2}\right)^2$$

$$y^2 = \frac{-2x^4 + x^4}{4}$$

$$4y^2 = -x^4$$

$$4y^2 + x^4 = 0$$

7. Solve the $y^2(y - px) = x^4 p^2$ by reducing it is in Clariat's form, taking the substitution $X = \frac{1}{x}, Y = \frac{1}{y}$

⇒ Given,

$$y^2(y - px) = x^4 p^2 \rightarrow ①$$

$$X = \frac{1}{x}, \quad Y = \frac{1}{y}$$

$$\Rightarrow \frac{dX}{dx} = -\frac{1}{x^2}, \quad \frac{dY}{dy} = -\frac{1}{y^2}$$

$$\frac{dy}{dx} = \frac{dy}{dx}\,\frac{dY}{dx}\,\frac{dX}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{\left(\frac{dY}{dy}\right)}\left(\frac{dY}{dX}\right)\left(\frac{dX}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{(-1/y^2)} P \cdot \left(-\frac{1}{x^2}\right)$$

$$p = \frac{Py^2}{x^2}$$

$$\Rightarrow p = \frac{P\,1/Y^2}{1/X^2}$$

$$\Rightarrow p = \frac{X^2}{Y^2} P$$

$$① \Rightarrow \frac{1}{Y^2}\left[\frac{1}{Y} - \frac{X^2}{Y^2} P \frac{1}{X}\right] = \frac{1}{X^4}\,\frac{X^4}{Y^4} P^2$$

$$\Rightarrow \frac{1}{y^2}\left[\frac{1}{x} - \frac{px}{y^2}\right] = \frac{p^2}{y^4}$$

$$\Rightarrow \frac{[y-px]}{y^4} = \frac{p^2}{y^4}$$

$$\Rightarrow y - px = p^2$$

$$\Rightarrow px + p^2 = y \longrightarrow (2)$$

$\therefore$ (2) is clariat's equation

$\therefore$ The Solution is,

$$y = cx + c^2$$

$$\frac{1}{y} = \frac{c}{x} + c^2 \;//$$

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Linear algebra.

* linear transformations

- A linear transformations in two dimension is defined as

$y_1 = a_1x_1 + a_2x_2$
$y_2 = b_1x_1 + b_2x_2$

The matrix form of the above transmission is given by

$Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ $A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

- Similarly a linear transformation in three dimension is defined as

$y_1 = a_1x_1 + a_2x_2 + a_3x_3$
$y_2 = b_1x_1 + b_2x_2 + b_3x_3$; $y_3 = c_1x_1 + c_2x_2 + c_3x_3$.

The matrix form is.

$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

- If $Y = AX$ representes a linear transformation than $X = A^{-1}Y$ representes inverse linear transformation.
- A linear transformation $Y = AX$ is said to Regular or Non-Singular if $|A| \neq 0$

1. Show that the linear transformation

$y_1 = 2x_1 + x_2 + x_3$
$y_2 = x_1 + x_2 + x_3$
$y_3 = x_1 - 2x_3$ is regular and hence find its inverse.

Given $y_1 = 2x_1 + x_2 + x_3$
$y_2 = x_1 + x_2 + 2x_3$
$y_3 = x_1 - 2x_3$

the matrix form of a LT is
$Y = AX$

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

(1) To p.T L.T is regular.

$$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{vmatrix} = -1 \neq 0$$

The L.T is regular (or) Non-Singular.

(2) To find Inverse L.T:-

$$A^{-1} = \frac{adj(A)}{|A|} = \begin{vmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{vmatrix}$$

hence $X = A^{-1}Y$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$x_1 = 2y_1 - 2y_2 - y_3$
$x_2 = -4y_1 + 5y + 3y_3$
$x_3 = y_1 - y_2 - y_3$.

Q. Show that the following linear transformation is non similar and find its inverse.

$y_1 = 2x_1 - 2x_2 - x_3$

$y_2 = -4x_1 + 5x_2 + 3x_3$

$y_3 = x_1 - x_2 - x_3$.

The matrix form of a LT is.

$Y = AX$ where

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \quad A = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

① To P.T L.T is regular

$$|A| = \begin{vmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{vmatrix} = -1 \neq 0.$$

the L.T is regular (or) Non-Singular.

② To find inverse L.T

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix}$$

hence $X = A^{-1}Y$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$x_1 = 2y_1 + y_2 + y_3$

$x_2 = y_1 + y_2 + 2y_3$

$x_3 = y_1 - 2y_3$.

3. Show that the following linear transformation is non similar and find its inverse

$y_1 = 8x_1 - 6x_2 + 2x_3$

$y_2 = -6x_1 + 7x_2 - 4x_3$

$y_3 = 2x_1 - 4x_2 + 3x_3$

The matrix form of a LT is

$Y = AX$ where

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

① To P.T L.T is regular

$$|A| = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix} = -40 \neq 0$$

The L.T is regular (or) Non-Singular

② To find Inverse L.T

$$\begin{pmatrix} 0.025 & -0.25 & -0.35 \\ -0.25 & -0.5 & -0.5 \\ -0.35 & -0.5 & -0.1 \end{pmatrix}$$

hence $X = A^{-1}Y$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0.025 & -0.25 & -0.35 \\ -0.25 & -0.5 & -0.5 \\ -0.35 & -0.5 & -0.1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$x_1 = 0.025y_1 - 0.25y_2 - 0.35y_3$

$x_2 = -0.25y_1 - 0.5y_2 - 0.5y_3$

$x_3 = -0.35y_1 - 0.5y_2 - 0.1y_3.$

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* Method of finding largest / Dominent Eigen value and Eigen vector

*** Rayleigh - Power

Working rule

(01) Step1 :- Given a square matrix A we choose the initial the eigen vector X_0 in the forms like $(1,0,0)^T$ (or) $[0,1,0]^T$ (or) $[0,0,0]^T$ (or) $[1,1,1]^T$

02. Step2 we compute AX_0, which is being a column matrix. In is matrix we take the largest element as a common factor and we write $AX_0 = \lambda_1 X_1$. This process is called as normallisation.

example
$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \underset{\lambda_1}{2} \begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix}_{X_1} = \lambda_1 X_1$$
(2 → highest)

03. Step3 we compute AX_1 compute AX_1. By the process normallisation we write $AX_1 = \lambda_2 X_2$

04. Step4 we repeat the above procedure till we get the root of desired accuracy.

***01. Find the largest (or) Dominent eigen value and eigen vector of the following matries by Raleigh - Power method.

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$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Let $X_0 = (1,0,0)^T$ be the initial eigen vector.

$$\Rightarrow A \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \lambda_1 X_1$$

$$\lambda_1 = 2; X_1 = \begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix}$$

$$AX_1 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 0 \\ 2.0 \end{bmatrix} = 2.5 \begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix} = \lambda_2 X_2.$$

$$AX_2 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 2.8 \\ 0 \\ 2.6 \end{bmatrix} = 2.8 \begin{bmatrix} 1 \\ 0 \\ 0.9285 \end{bmatrix} = \lambda_3 X_3.$$

0.9285.

$$AX_3 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.9285 \end{bmatrix} = \begin{bmatrix} 2.9285 \\ 0 \\ 2.857 \end{bmatrix} = 2.9285 \begin{bmatrix} 1 \\ 0 \\ 0.9755 \end{bmatrix} = \lambda_4 X_4.$$

$$AX_4 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.9755 \end{bmatrix} = \begin{bmatrix} 2.9755 \\ 0 \\ 2.955 \end{bmatrix} = 2.9755 \begin{bmatrix} 1 \\ 0 \\ 0.9931 \end{bmatrix} = \lambda_5 X_5.$$

$$AX_5 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.9931 \end{bmatrix} = \begin{bmatrix} 2.9931 \\ 0 \\ 2.9862 \end{bmatrix} = 2.9931 \begin{bmatrix} 1 \\ 0 \\ 0.9976 \end{bmatrix} = \lambda_6 X_6$$

$$\lambda = 2.9931 \approx 3$$
$$X = (1, 0, 0.9976)^T \approx [1, 0, 1]^T$$

Q2. $A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$ by taking $(1, 0.5, 0.5)^T$ as inital eigen vector.

$x_0 = (1, 0.5, 0.5)^T$ be the initial eigen vector.

$$AX_0 = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ -4.5 \end{bmatrix} = -4.5 \begin{bmatrix} -0.5 \\ 0.6 \\ -1 \end{bmatrix} = \lambda_1 X_1$$

$$AX_1 = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix} \begin{bmatrix} -0.5 \\ 0.6 \\ -1 \end{bmatrix} = \begin{bmatrix} -0.1 \\ 0.4 \\ -0.6 \end{bmatrix} = 0.6 \begin{bmatrix} -0.67 \\ 0.67 \\ -1 \end{bmatrix} = \lambda_2 X_2$$

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$$AX_2 = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix} \begin{bmatrix} -0.67 \\ 0.67 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.696 \\ 0.68 \\ -1.02 \end{bmatrix} = 1.02 \begin{bmatrix} -0.676 \\ 0.67 \\ -1 \end{bmatrix} = \lambda_3 X_3$$

$$AX_3 = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix} \begin{bmatrix} -0.676 \\ 0.67 \\ -1 \end{bmatrix} = \begin{bmatrix} -0.696 \\ 0.68 \\ -1.02 \end{bmatrix} = 1.02 \begin{bmatrix} -0.688 \\ 0.68 \\ -1 \end{bmatrix} \lambda_4 X_4.$$

$$\lambda = 1.02 \approx 1$$

$$X = [0.688, 0.68, -1]$$

3. $$A = \begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix}$$ by taking $x_0 = (1, 1, 1)^T$

X_0 is $(1,1,1)^T$.

$$AX_0 \begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix} \begin{bmatrix} 1 \\ 0.8 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 2.2 \\ 1.4 \\ 2 \end{bmatrix} = 2.2 \begin{bmatrix} 1 \\ 0.6363 \\ 0.9090 \end{bmatrix}$$

$$AX_1 = \begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix} \begin{bmatrix} 1 \\ 0.6363 \\ 0.9090 \end{bmatrix} = \begin{bmatrix} 2.0918 \\ 1.1824 \\ 2.0008 \end{bmatrix} = 2.0918 \begin{bmatrix} 1 \\ 0.565 \\ 0.956 \end{bmatrix} = \lambda_2 X_2.$$

$$AX_2 = \begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix} \begin{bmatrix} 1 \\ 0.565 \\ 0.956 \end{bmatrix} = \begin{bmatrix} 2.048 \\ 1.09 \\ 2.004 \end{bmatrix} = 2.048 \begin{bmatrix} 1 \\ 0.532 \\ 0.978 \end{bmatrix} = \lambda_3 X_3$$

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$$AX_3 \begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix} \begin{bmatrix} 1 \\ 0.532 \\ 0.978 \end{bmatrix} = \begin{bmatrix} 2.026 \\ 1.046 \\ 2.004 \end{bmatrix} = 2.026 \begin{bmatrix} 1 \\ 0.516 \\ 0.989 \end{bmatrix} = \lambda_4 X_4$$

$\lambda = 2.026 \simeq 2$

$x = (1, 0.516, 0.989)^T \simeq (1, 0.5, 1)^T$

4. $A \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

x^0 is $[1,0,0]^T$.

$$AX_0 = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ -6 \\ 2 \end{bmatrix} = 8 \begin{bmatrix} 1 \\ -0.75 \\ 0.25 \end{bmatrix} = \lambda_1 X_1$$

$$AX_1 = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -0.75 \\ 0.25 \end{bmatrix} = \begin{bmatrix} 13 \\ -12.25 \\ 5.75 \end{bmatrix} = 13 \begin{bmatrix} 1 \\ -0.942 \\ 0.442 \end{bmatrix} = \lambda_2 X_2.$$

$$AX_2 = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -0.942 \\ 0.442 \end{bmatrix} = \begin{bmatrix} 14.536 \\ -14.362 \\ 7.094 \end{bmatrix} = 14.536 \begin{bmatrix} 1 \\ -0.988 \\ 0.488 \end{bmatrix} = \lambda_3 X_3$$

$$AX_3 = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -0.988 \\ 0.488 \end{bmatrix} = \begin{bmatrix} 14.904 \\ -14.868 \\ 7.416 \end{bmatrix} = 14.904 \begin{bmatrix} 1 \\ -0.997 \\ 0.497 \end{bmatrix} = \lambda_4 X_4.$$

$$AX_4 = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -0.997 \\ 0.497 \end{bmatrix} = \begin{bmatrix} 14.976 \\ -14.967 \\ 7.479 \end{bmatrix} = 14.976 \begin{bmatrix} 1 \\ -0.999 \\ 0.499 \end{bmatrix}$$

$\lambda = 14.976 \simeq 15$ $\quad x = (1, -0.999, 0.499)^T \simeq (1, -1, 0.5)^T$

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* Rank of martix

* Row reduce Echelon form
A matrix 'A' is said to in row reduce Echelon form if it satifies the following conditions.

01. The leading entre of each row must be non-zero element.
02. The element below the leading entre are zeroes.
03. The number of zeroes in each row must be greater than its previous row
04. If there is a zero row it should be write below the non-zero rows.

• In other word Echelon form martix represents an upper triangular martix. (element below the prinicple diagonal are zero).

* Rank
The Rank of matrix 'A' denoted by $\rho(A)$ representes the no. of non-zero rows in Echelon form matrix.

Q. problems
1. Find the rank of following matries appling elementary row transmition or by reducing them to row reduced enchlen form.

(i) $A \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 3 & 5 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix}$

(iii) $A \begin{bmatrix} 4 & 0 & 2 & 1 \\ 2 & 1 & 3 & 4 \\ 2 & 3 & 4 & 7 \\ 2 & 3 & 1 & 4 \end{bmatrix}$ (iv) $A \begin{bmatrix} 2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

(v) $\begin{bmatrix} 2 & -4 & 3 & 10 & 0 \\ 1 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 0 \end{bmatrix}$

(i) $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

In = $R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - R_1$
$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $R_3 \to R_3 + R_2$.

$\rho(A) = 2$ //.

2(A) = $\begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 3 & 5 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix}$ $R_1 \leftrightarrow R_3$.

$\begin{bmatrix} 4 & 8 & 13 & 12 \\ 2 & 3 & 5 & 4 \\ 0 & 2 & 3 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 8 & 13 & 12 \\ 0 & -2 & -3 & -4 \\ 0 & 2 & 3 & 4. \end{bmatrix}$ $R_2 \to 2R_2 - R_1$

$\begin{bmatrix} 4 & 8 & 13 & 12 \\ 0 & -2 & -3 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $R_3 \to R_3 + R_2$

$\rho(A) = 2$ //.

3 $A = \begin{bmatrix} 4 & 0 & 2 & 1 \\ 2 & 1 & 3 & 4 \\ 2 & 3 & 4 & 7 \\ 2 & 3 & 1 & 4 \end{bmatrix}$

$n = \begin{bmatrix} 4 & 0 & 2 & 1 \\ 0 & 2 & 4 & 7 \\ 0 & 6 & 6 & 13 \\ 0 & 6 & 0 & 7 \end{bmatrix}$ $R_2 \to 2R_2 - R_1$, $R_3 \to 2R_3 - R_1$, $R_4 \to 2R_4 - R_1$

$n = \begin{bmatrix} 4 & 0 & 2 & 1 \\ 0 & 2 & 4 & 7 \\ 0 & 0 & -6 & -8 \\ 0 & 0 & -12 & -14 \end{bmatrix}$ $R_3 \to R_3 - 3R_2$, $R_4 \to R_4 - 3R_2$

$n = \begin{bmatrix} 4 & 0 & 2 & 1 \\ 0 & 2 & 4 & 7 \\ 0 & 0 & -6 & -8 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ $R_4 \to R_4 - 2R_3$

$\rho(A) = 4$ //.

4 = $A = \begin{bmatrix} 2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

$n = \begin{bmatrix} 2 & -1 & -3 & -1 \\ 0 & 5 & 9 & -1 \\ 0 & 1 & 5 & 3 \\ 0 & 1 & 1 & -1 \end{bmatrix}$ $R_2 \to 2R_2 - R_1$, $R_3 \to 2R_3 - R_1$

$$n=\begin{bmatrix} 2 & -1 & -3 & -1 \\ 0 & 5 & 9 & -1 \\ 0 & 5 & 16 & 16 \\ 0 & 0 & -4 & -4 \end{bmatrix} \quad \begin{matrix} R_3 \to 5R_3 - R_2 \\ R_4 \to 5R_4 - R_2 \end{matrix}$$

$$n=\begin{bmatrix} 2 & -1 & -3 & -1 \\ 0 & 5 & 9 & -1 \\ 0 & 0 & 16 & 16 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_4 \to 4R_4 + R_3.$$

$$f(A) = 3.$$

(5)
$$\begin{bmatrix} 2 & -4 & 3 & 1 & 0 \\ 1 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 0 \end{bmatrix}$$

$-4+8$

$$n=\begin{bmatrix} 2 & -4 & 3 & 1 & 0 \\ 0 & 0 & -1 & -9 & 4 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 1 & -2 & +4 & -4 \end{bmatrix} \quad \begin{matrix} R_2 \to 2R_2 - R_1 \\ R_4 \to R_4 - 2R_1 \end{matrix}$$

$$n=\begin{bmatrix} 2 & -4 & 3 & 1 & 0 \\ 0 & 1 & -2 & +4 & -4 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & -1 & -9 & 4 \end{bmatrix} \quad R_2 \leftrightarrow R_4.$$

$-1+2$

$$n=\begin{bmatrix} 2 & -4 & 3 & 1 & 0 \\ 0 & 1 & -2 & +4 & -4 \\ 0 & 0 & +1 & -1 & 5 \\ 0 & 0 & -1 & -9 & 4. \end{bmatrix} \quad R_3 \to R_3 - R_2$$

$$n=\begin{bmatrix} 2 & -4 & 3 & 1 & 0 \\ 0 & 1 & 2 & -4 & -4 \\ 0 & 0 & -3 & 5 & 5 \\ 0 & 0 & 0 & -32 & 7 \end{bmatrix} \quad R_4 \to 3R_4 + R_3$$

$\rho(A) = 4$

$$n=\begin{bmatrix} 2 & -4 & 3 & 1 & 0 \\ 0 & 1 & -2 & +4 & 4 \\ 0 & 0 & +1 & -1 & 5 \\ 0 & 0 & -1 & -9 & 4 \end{bmatrix}$$

$$n=\begin{bmatrix} 2 & -4 & 3 & 1 & 0 \\ 0 & 1 & -2 & 4 & 4 \\ 0 & 0 & 1 & -1 & 5 \\ 0 & 0 & 0 & -10 & 9 \end{bmatrix} \quad R_4 \to R_4 + R_3$$

$\rho(A) = 4.$

* Solution of linear, simultaneous equation

(I) Gauss- Elimination method

- Step-1 Given a system of equation, we write the Augmented matrix [A : B]
- Step-2 We apply eliminator row elementary operation we reduce augmented matrix into row reduce echelon form.
- Step3 We write the equation for the echelon form matrices. and solving. we get the required solution.

Q1 Solve the following system of equation using Gauss-elimination method.

(i) $x+y+z=9$

$x-2y+3z=8$

$2x+y-z=3.$

The augmented matrix is.

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : 9 \\ 1 & -2 & 3 & : 8 \\ 2 & 1 & -1 & : 3 \end{bmatrix}$$

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$$I_n = \begin{bmatrix} 1 & 1 & 1 & : 9 \\ 0 & -3 & 2 & : -1 \\ 0 & -1 & -3 & : -15 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$II_n = \begin{bmatrix} 1 & 1 & 1 & : 9 \\ 0 & -3 & 2 & : -1 \\ 0 & 0 & -11 & : -44 \end{bmatrix} \quad R_3 \to 3R_3 - R_2$$

$x + y + z = 9$

$-3y + 2z = -1$

$-11z = -44.$

$x = 2, \; y = 3, \; z = 4.$

(2) Given $2x_1 + x_2 + 4x_3 = 12$

$4x_1 + 11x_2 - x_3 = 33$

$8x_1 - 3x_2 + 2x_3 = 20.$

The augmented matrix is.

$$(A:B) \begin{bmatrix} 2 & 1 & 4 & : 12 \\ 4 & 11 & -1 & : 33 \\ 8 & -3 & 2 & : 20 \end{bmatrix}$$

$$I_n = \begin{bmatrix} 2 & 1 & 4 & : 12 \\ 0 & 9 & -9 & : 9 \\ 0 & -7 & 14 & : -26 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1 \end{matrix}$$

$$II_n = \begin{bmatrix} 2 & 1 & 4 & : 12 \\ 0 & 9 & -9 & : 9 \\ 0 & 0 & -189 & : -189 \end{bmatrix} \quad R_3 \to 9R_3 + 7R_2$$

$2x + y + 4z = 12.$

$9y - 9z = 9,$

$-189z = -189.$

Solving we get.

$x = 3$

$y = 2$

$z = 1$

(iii) Given

$10x+y+z=12$

$2x+10y+z=13$

$x+y+5z=7$

The augmented matrix is.

$$(A:B)=\begin{bmatrix} 10 & 1 & 1 & : 12 \\ 2 & 10 & 1 & : 13 \\ 1 & 1 & 5 & : 7 \end{bmatrix}$$

$$\text{Rn}=\begin{bmatrix} 10 & 1 & 1 & : 12 \\ 0 & 49 & 4 & : 53 \\ 0 & 9 & 49 & : 58 \end{bmatrix} \begin{matrix} \\ R_2 \to 5R_2 - R_1 \\ R_3 \to 10R_3 - R_1 \end{matrix}$$

$$\text{Rn}\begin{bmatrix} 10 & 1 & 1 & : 12 \\ 0 & 49 & 4 & : 53 \\ 0 & 0 & 2365 & : 2365 \end{bmatrix} R_3 \to 49R_3 - 9R_2.$$

$10x+y+z=12$

$49y+4z=53$

$2365z=2365$

solving we get

$x=1, y=1, z=1.$

(iv) $5x_1+x_2+x_3+x_4=4.$

$x_1+7x_2+x_3+x_4=12$

$x_1+x_2+6x_3+x_4=-5$

$x_1+x_2+x_3+4x_4=-6$

-2, -1, 1, 2

The augmented matrix is

$$(A:B)=\begin{bmatrix} 5 & 1 & 1 & 1 & : 4 \\ 1 & 7 & 1 & 1 & : 12 \\ 1 & 1 & 6 & 1 & : -5 \\ 1 & 1 & 1 & 4 & : -6 \end{bmatrix}$$

$$\text{Rn}=\begin{bmatrix} 5 & 1 & 1 & 1 & : 4 \\ 0 & 34 & 4 & 4 & : 54 \\ 0 & 4 & 29 & 4 & : -29 \\ 0 & 4 & 4 & 19 & : -34 \end{bmatrix} \begin{matrix} \\ R_2 \to 5R_2 - \\ R_3 \to 5R_3 - \\ R_4 \to 5R_4 - \end{matrix}$$

$$\sim \begin{bmatrix} 5 & 1 & 1 & 1 & : 4 \\ 0 & 34 & 4 & 4 & : 54 \\ 0 & 0 & & & \\ 0 & 0 & & & \end{bmatrix} \quad \begin{matrix} R_3 \to 34R_3 - 4R_2 \\ R_4 \to 34R_4 - 4R_2 \end{matrix}$$

Gauss-Jordan Method

working rule.

Step 1. Give a system of equations we write the augmented matrix

Step 2. we apply the elementary row operation and reduce into Echelon form

Step 3. again by applying row operation we the matrix into diagonal form.

Step 4. we write the equations for diagonal form matrix and solving we get the required solution.

Q.1 Solve the following system by Gauss-Jordan Method.

(i) $x+y+z=9$

$x-2y+3z=8$

$2x+y-z=3$.

The augmented matrix is.

$$[A:B] = \begin{bmatrix} 1 & 1 & 1 & : 9 \\ 1 & -2 & 3 & : 8 \\ 2 & 1 & -1 & : 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & : 9 \\ 0 & -3 & 2 & : -1 \\ 0 & -1 & -3 & : -15 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$II_n = \begin{bmatrix} 1 & 1 & 1 & : & 9 \\ 0 & -3 & 2 & : & -1 \\ 0 & 0 & -11 & : & -44 \end{bmatrix} \quad R_3 \to 3R_3 - R_2.$$

$$\sim \begin{bmatrix} 11 & 11 & 0 & : & 55 \\ 0 & -33 & 0 & : & -99 \\ 0 & 0 & -11 & : & -44 \end{bmatrix} \quad \begin{matrix} R_1 \to 11R_1 + R_3. \\ R_2 \to 11R_2 + 2R_3. \end{matrix}$$

$$\sim \begin{bmatrix} 33 & 0 & 0 & : & 66 \\ 0 & -33 & 0 & : & -99 \\ 0 & 0 & -11 & : & -44 \end{bmatrix} \quad R_1 \to 3R_1 + R_2.$$

$33x = 66 \quad x = 2$

$-33y = -99 \quad y = 3.$

$-11z = -44 \quad z = 4.$

(ii) $2x_1 + x_2 + 4x_3 = 12$

$4x_1 + 11x_2 - x_3 = 33$

$8x_1 - 3x_2 + 2x_3 = 20$

The augmented matrix is

$$[A:B] = \begin{bmatrix} 2 & 1 & 4 & : & 12 \\ 4 & 11 & -1 & : & 33 \\ 8 & -3 & 2 & : & 20 \end{bmatrix}$$

$$I_n = \begin{bmatrix} 2 & 1 & 4 & : & 12 \\ 0 & 9 & -9 & : & 9 \\ 0 & -7 & -14 & : & -28 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1 \end{matrix}$$

$$II_n = \begin{bmatrix} 2 & 1 & 4 & : & 12 \\ 0 & 9 & -9 & : & 9 \\ 0 & 0 & -189 & : & -189 \end{bmatrix} \quad R_3 \to 9R_3 + 7R_2$$

$R_2/9 - R_3/-189$

$$\sim \begin{bmatrix} 2 & 1 & 4 & : & 12 \\ 0 & 1 & -1 & : & 1 \\ 0 & 0 & 0 & : & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 1 & 0 & : & 8 \\ 0 & 1 & 0 & : & 2 \\ 0 & 0 & 1 & : & 1 \end{bmatrix} \quad \begin{matrix} R_1 \to R_1 - 4R_3 \\ R_2 \to R_2 + R_3. \end{matrix}$$

$$\sim \begin{bmatrix} 2 & 0 & 0 & : & 6 \\ 0 & 1 & 0 & : & 2 \\ 0 & 0 & 1 & : & 1 \end{bmatrix} \quad R_1 \to R_1 - R_2$$

$2x = 6 \Rightarrow x = 3$

$y = 2$

$z = 1$

(iii) $10x + y + z = 12$

$2x + 10y + z = 13$

$x + y + 5z = 7$

The augmented matrix is

$$[A:B] = \begin{bmatrix} 10 & 1 & 1 & : & 12 \\ 2 & 10 & 1 & : & 13 \\ 1 & 1 & 5 & : & 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 10 & 1 & 1 & : & 12 \\ 0 & 49 & 4 & : & 53 \\ 0 & 9 & 49 & : & 58 \end{bmatrix} \quad \begin{matrix} R_2 \to 5R_2 - R_1 \\ R_3 \to 10R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 10 & 1 & 1 & : & 12 \\ 0 & 49 & 4 & : & 53 \\ 0 & 0 & 2365 & : & 2365 \end{bmatrix} \quad R_3 \to 49R_3 - 9R_2$$

$R_3 / 2365$

$$\sim \left[\begin{array}{ccc|c} 10 & 1 & 1 & 12 \\ 0 & 49 & 4 & 53 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 10 & 1 & 0 & 11 \\ 0 & 49 & 0 & 49 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - R_3 \\ R_2 \to R_2 - 4R_3. \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 490 & 0 & 0 & 490 \\ 0 & 49 & 0 & 49 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad R_1 \to 49R_1 - R_2.$$

$490x = 490 \Rightarrow x = 1$

$49y = 49 \Rightarrow y = 1$

$z = 1$ //.

7/11/2017

** Gauss-Siedel Iterative method

Suppose we have a system of linear simealtaneous equations

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$. such that

$|a_{11}| > |a_{12}| + |a_{13}|$

$|a_{22}| > |a_{21}| + |a_{23}|$

$|a_{33}| > |a_{31}| + |a_{32}|$ then the system is said to be in diagonal dominat form and hence we can apply Gauss-Siedel Method as follows

- If the system of eqaation are not in diagonal dominat form then we rearrange the equations, we express them as dominate system and hence we apply gass-seidel method.

Q.1 Solve the following system of equation using Gauss-siedel iterative method.

(i) $10x + y + z = 12$ $\quad x = 1/10(12 - y - z)$

$x + 10y + z = 12$ $\quad y = 1/10(12 - x - z)$

$x + y + 10z = 12.$ $\quad z = 1/10(12 - x - y)$

let $(x, y, z) = (0, 0, 0)$ be the trail solution or initial approximation.

I^st app :- $(x)' = \frac{1}{10}(12 - 0 - 0) = 1.2$

$(y)' = 1/10(12 - 1.2 - 0) = 1.08$

$(z)' = 1/10(12 - 1.2 - 1.08) = 0.972.$

II^nd app = $(x)'' = 1/10(12 - 1.08 - 0.972) = 0.9948$

$(y)'' = 1/10(12 - 0.9948 - 0.972) = 1.0033$

$(z)^2 = 1/10(12 - 0.9948 - 1.0033) = 1.0002.$

III^rd app = $(x)''' = 1/10(12 - 1.0033 - 1.0002) = 0.9997$

$(y)''' = 1/10(12 - 0.9997 - 1.0002) = 1.0000$

$(z)''' = 1/10(12 - 0.9997 - 1.0000) = 1.0000$

$(x, y, z) = (1, 1, 1).$

(ii) $5x + 2y + z = 12$

$x + 4y + 2z = 15$

$x + 2y + 5z = 20$

by taking (1, 0, 3) as initial approximation.

$x = 1/5(12 - 2y - z)$

$y = 1/4(15 - x - 2z)$

$z = 1/5(20 - x - 2y)$

let $(x, y, z) = (1, 0, 3)$ be the initial approximation.

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1st app. $(x)' = 1/5[12-0-3] = 1.8$

$(y)' = 1/4[15-1.8-(2\times3)] = 1.8$

$(z)' = 1/5[20-1.8-(2\times1.8)] = 2.95.$

2nd app $(x)'' = 1/5[12-1.8^{\times2}-2.95] = 1.096.$

$(y)'' = 1/4(12-1.096-2.95^{\times2}) = 2.016$

$(z)'' = 1/5(12-1.096-2.016^{\times2}) = 2.9744.$

III app $(x)''' = 1/5[12-(2\times2.016)-2.9744] = 0.9987$

$(y)''' = 1/4[15-0.9987-(2\times2.9744)] = 2.0131$

$(z)''' = (1/5)[20-0.9987-(2\times2.0131)] = 2.9950.$

$(x,y,z) = (1,2,3).$

3. $x+y+54z=110$

$27x+6y-z=85.$

$6x+15y+2z=72.$

Here, the given eqn are not in diagonally dominant form Hence, we rearrange the eqn as follows.

$27x+6y-z=85$

$6x+15y+2z=72$

$x+y+54z=110.$

$x = 1/27[85-6y+z]$

$y = 1/15[72-6x-2z]$

$z = 1/54[110-x-y]$

let $(x,y,z) = (0,0,0)$ be the trail solution.

1st app $(x)' = 1/27(85-0+0) = 3.1481$

$(y)' = 1/15(72-(6\times3.1481)-0) = 3.5408$

$(z)' = 1/54[110-3.1481-3.5408] = 1.9132.$

2nd app $(x)^2 = 1/27(85 - (6\times3.5408) + 1.9132) = 2.4322$

$(y)^2 = 1/15[72 - (6\times2.4322) - (2\times1.9132)] = 3.5720$

$(z)^2 = 1/54[110 - 2.4322 - 3.5720) = 1.9258$

III rd app $(x)^3 = 1/27[85 - (6\times3.5720) + 1.9258] = 2.4256$

$(y)^3 = 1/15(72 - (6\times2.4856) - 2(1.9258)) = 3.5729$

$(z)^3 = 1/54(110 - 2.4256 - 3.5729) = 1.9259$

$(x,y,z) = (2.42, 3.57, 1.92)$

9/11/2021

* Eigen values and Eigen vectors.

working rule

Step 1 Give a square matrix A, we write the characterist equation. $|A-\lambda I|=0$

Step 2 To expand the determinate we take a polynomial in λ, solving the polynomial we get. the roots which are called as Eigen values.

Step 3 for each Eigen value we solve the system of equations obtained from the characterist martix we get the Eigen vectors.

Q1. Find the Eigen values and corresponding Eigen vectors of the following martices.

(i) $A = \begin{bmatrix} -3 & 8 \\ -2 & 7 \end{bmatrix}$ (ii) $A = \begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}$ (iii) $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

(iv) $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ (v) $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

(ii) $A = \begin{bmatrix} -3 & 8 \\ -2 & 7 \end{bmatrix}$

The ch. eq^n is $|A - \lambda I| = 0$

$$\begin{vmatrix} -3-\lambda & 8 \\ -2 & 7-\lambda \end{vmatrix} = 0$$

$(-3-\lambda)(7-\lambda) + 16 = 0$

$-21 + 3\lambda - 7\lambda + \lambda^2 + 16 = 0$

$\lambda^2 - 4\lambda - 5 = 0$

$\lambda = 5$ & -1 are the eigen value.

consider $\begin{bmatrix} -3-\lambda & 8 \\ -2 & 7-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\left.\begin{aligned} (-3-\lambda)x_1 + 8x_2 = 0 \\ -2x_1 + (7-\lambda)x_2 = 0 \end{aligned}\right\} \Rightarrow (1)$$

case(i) for $\lambda = 5$, eq^n $\Rightarrow$ (1)

$-8x_1 + 8x_2 = 0 \Rightarrow 8x_1 = 8x_2 = x_1 = x_2$

$-2x + 2x_2 = 0 \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$

let $x_2 = 1$ $\quad x_1 = 1$

$(x_1, x_2)^T = (1, 1)^T$

case(ii) for $\lambda = 1$ (1) $\Rightarrow$

$-2x_1 + 8x_2 = 0 \Rightarrow 2x_1 = 8x_2 = x_1 = 4x_2$

$-2x_1 + 8x_2 = 0 \Rightarrow 2x_1 = 8x_2 = x_1 = 4x_2$

let $x_2 = 1$ $\therefore x_1 = 4$.

$(x_1, x_2)^T = (4, 1)^T$

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(ii) $A = \begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}$

The ch. eq^n is $|A - \lambda I| = 0$

$$\begin{vmatrix} -1-\lambda & 3 \\ -2 & 4-\lambda \end{vmatrix} = 0.$$

$(-1-\lambda)(4-\lambda) + 6 = 0$

$-4 + \lambda - 4\lambda + \lambda^2 + 6 = 0$

$\lambda^2 - 3\lambda + 2 = 0$

$\lambda = 2$ & 1. are the eigen value.

consider.

$$\begin{bmatrix} -1-\lambda & 3 \\ -2 & 4-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(-1-\lambda)x_1 + 3(x_2) = 0$

$-2x_1 + (4-\lambda)x_2 = 0$

case 1 for $\lambda = 2$.

$-3x_1 + 3x_2 = 0 \Rightarrow 3x_1 = 3x_2 = x_1 = x_2$

$-2x_1 + 2x_2 = 0 \Rightarrow 2x_1 = 2x_2 = x_1 = x_2$.

let $x_2 = 1 \therefore x_1 = 1$

$(x_1, x_2)^T = (1, 1)^T$

case 2 for $\lambda = 1$

$-2x_1 + 3x_2 = 0 \quad 2x_1 = 3x_2 = x_1 = 3/2 x_2$

$-2x + 3x_2 = 0 \quad 2x_1 = 3x_2 = x_1 = 3/2 x_2$

let $x_2 = 1 \quad x_1 = 3/2$.

$(x_1, x_2)^T = (3/2, 1)^T$

(iii) $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

$(2-\lambda)\left[4-2\lambda-2\lambda+\lambda^2\right]$

$8-4\lambda-4\lambda+2\lambda^2-4\lambda+2\lambda^2+2\lambda^2-\lambda^3$

$-2+\lambda$

~~8~~ $-$ ~~8λ~~ $+2\lambda^2-4\lambda+2\lambda^2+2\lambda^2-\lambda^3-$ ~~$2+\lambda$~~

$6-7\lambda+6\lambda^2-\lambda^3$

$6-11\lambda+6\lambda^2-\lambda^3.$

$-\lambda^3+6\lambda^2-11\lambda+6.$

The ch. eq^n is $|A-\lambda I|=0$

$$\begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0.$$

~~$(2-\lambda)\left[(2-\lambda)(2-\lambda)\right]+1\left[-(2-\lambda)(1)\right].$~~

~~$(2-\lambda)\left[4-2\lambda+2\lambda+\lambda^2\right]+1\left[-2+\lambda\right]$~~

~~$8-4\lambda-4\lambda-2\lambda^2-4\lambda+2\lambda^2+2\lambda^2+\lambda^3-2+\lambda$~~

~~$8-8\lambda-\ 8-12\lambda+2\lambda^2+\lambda^3-2+\lambda$~~

~~$6-11\lambda+2\lambda^2+\lambda^3$~~

~~$\lambda^3+2\lambda^2-11\lambda+6=0$~~

$(2-\lambda)^3-(2-\lambda)=0$

$(2-\lambda)\left[(2-\lambda)^2-1\right]=0$

$2-\lambda=0$

$\lambda=2$

$4+\lambda^2-4\lambda-1=0$

$\lambda^2-4\lambda+3=0$

$\lambda=3,1.$

$\lambda = 1, 2, 3.$ are the egigen values.

consider.

$$\begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\left.\begin{aligned} (2-\lambda)x_1 + x_3 &= 0 \\ (2-\lambda)x_2 &= 0 \\ x_1 + (2-\lambda)x_3 &= 0 \end{aligned}\right\} \text{①}$$

for (i) $\lambda = 1.$

$x_1 + x_3 = 0 \Rightarrow x_1 = -x_3$

$x_2 = 0$

$x_1 + x_3 = 0 \Rightarrow x_1 = -x_3$

let $x_3 = 1$ $\therefore x_1 = -1$

$(x_1, x_2, x_3)^T = (-1, 0, 1)^T$

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for $\lambda = 2$.

$x_3 = 0$

$x_1 = 0$ let $x_2 = 1$ because if it becomes $(0,0,0)$ it is non zero

$(x_1, x_2, x_3)^T = (0, 1, 0)^T$

for $\lambda = 3$.

$-1x_1 + x_3 = 0 \Rightarrow -x_1 + x_3 = 0 \quad x_1 = x_3$

$-1x_2 = 0 \Rightarrow -x_2 = 0 \quad +x_2 = 0$

$x_1 - 1x_3 = 0. \Rightarrow x_1 - x_3 = 0 \quad x_1 = x_3$

let $x_3 = 1 \therefore x_1 = 1$

$(x_1, x_2, x_3)^T = (1, 0, 1)^T$ //.

6(A) $= \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

The ch eq^n is $(A - \lambda I) = 0$.

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0.$$

$(8-\lambda)[(7-\lambda)(3-\lambda) - (-4)(-4)] + 6[-6(3-\lambda) + 8] + 2[24 - 2(7-\lambda)] = 0$

$(8-\lambda)[21 - 3\lambda - 7\lambda + \lambda^2 - 16] + 6[-18 + 6\lambda + 8] + 2(24 - 12 + 2\lambda)$

$8-\lambda[\lambda^2 - 10\lambda + 5] + 36\lambda - 60 + 4\lambda + 20 = 0$

$8\lambda^2 - 80\lambda + 40 - \lambda^3 + 10\lambda^2 - 5\lambda + 40\lambda - 40 = 0$

$-\lambda^3 + 18\lambda^2 - 45\lambda = 0$

$\lambda = 15, 3, 0$ are the eigen values.

$$\begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow \begin{matrix} (8-\lambda)x_1 - 6x_2 + 2x_3 = 0 \\ -6x_1 + (7-\lambda)x_2 - 4x_3 = 0 \\ 2x_1 - 4x_2 + (3-\lambda)x_3 = 0 \end{matrix}$$

case 1.

For $\lambda = 0 \Rightarrow$.

$8x_1 - 6x_2 + 2x_3 = 0$

$-6x_1 + 7x_2 - 4x_3 = 0$

$2x_1 - 4x_2 + 3x_3 = 0$

case 2.

for $\lambda = 3$

$5x_1 - 6x_2 + 2x_3 = 0$

$-6x_1 + 4x_2 - 4x_3 = 0$

$2x_1 - 4x_2 = 0$

case 3

for $\lambda = 15$.

$-7x_1 - 6x_2 + 2x_3 = 0$

$-6x_1 - 8x_2 - 4x_3 = 0$

$2x_1 - 4x_2 - 12x_3 = 0$.

using RCM,

case 1 $\dfrac{x_1}{\begin{vmatrix} -6 & 2 \\ 7 & 4 \end{vmatrix}} = \dfrac{-x_2}{\begin{vmatrix} 8 & 2 \\ -6 & -4 \end{vmatrix}} = \dfrac{x_3}{\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}} = 1$

$\dfrac{x_1}{10} = \dfrac{+x_2}{+20} = \dfrac{x_3}{20} = 1 \Big) = \dfrac{x_1}{1} = \dfrac{x_2}{2} = \dfrac{x_3}{2}$.

$(x_1, x_2, x_3)^T = (1, 2, 2)^T$

case 2

$\dfrac{x_1}{\begin{vmatrix} -6 & 2 \\ 4 & -4 \end{vmatrix}} = \dfrac{-x_2}{\begin{vmatrix} 5 & 2 \\ -6 & -4 \end{vmatrix}} = \dfrac{x_3}{\begin{vmatrix} 5 & -6 \\ -6 & 4 \end{vmatrix}}$

$\dfrac{x_1}{16} = \dfrac{+x_2}{+8} = \dfrac{x_3}{-16} = 1$

$\dfrac{x_1}{2} = \dfrac{x_2}{1} = \dfrac{x_3}{-2} = 1$

$(x_1, x_2, x_3) = (2, 1, -2)^T$.

case 3 $\dfrac{x_1}{\begin{vmatrix} -6 & 2 \\ -8 & -4 \end{vmatrix}} = \dfrac{-x_2}{\begin{vmatrix} -7 & 2 \\ -6 & -4 \end{vmatrix}} = \dfrac{x_3}{\begin{vmatrix} -7 & -6 \\ -6 & -8 \end{vmatrix}}$

$\dfrac{x_1}{40} = \dfrac{-x_2}{40} = \dfrac{x_3}{20} = 1$

$\dfrac{x_1}{2} = \dfrac{x_2}{-2} = \dfrac{x_3}{1} = 1$

$(x_1, x_2, x_3) = (2, -2, 1)^T$ //.

V $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ The ch eqn is $|A - \lambda I| = 0$ $= \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} = 0$.

$(1-\lambda)[(5-\lambda)(1-\lambda)-1] - 1[3-(1-\lambda)] + 3[1-(5-\lambda)(3)]. = 0$

$(1-\lambda)[5 - 5\lambda - \lambda + \lambda^2 - 1] - 1[3 - 1 + \lambda] + 3[1 - 15 + 3\lambda]. = 0$

$(1-\lambda)[4 - 6\lambda + \lambda^2] - 1[2 + \lambda] + 3[-14 + 3\lambda] = 0$

$4 - 6\lambda + \lambda^2 - 4\lambda + 6\lambda^2 - \lambda^3 + 2 - \lambda - 42 + 9\lambda. = 0$

$-\lambda^3 - 10\lambda + 7\lambda^2 + 2 - \lambda + 40 + 9\lambda.$

$-\lambda^3 + 7\lambda^2 - 2\lambda + 44 = 0.$

$= -\lambda^3 + 7\lambda^2 - 36 = 0$

$\lambda = -2, 3$ and 6 are the eigen nvalues.

$$\begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{matrix} (1-\lambda)x_1 + x_2 + 3x_3 = 0 \\ x_1 + (5-\lambda)x_2 + x_3 = 0 \\ 3x_1 + x_2 + (1-\lambda)x_3 = 0. \end{matrix}$$

case 1 for = -2.

$3x_1 + x_2 + 3x_3 = 0$

$x_1 + 7x_2 + x_3 = 0$

$3x_1 + x_2 + 3x_3 = 0$

case 2 for = 3.

$-2x_1 + x_2 + 3x_3 = 0$

$x_1 + 2x_2 + x_3 = 0$

$3x_1 + x_2 - 2x_3 = 0.$

case 3 for 6.

$-5x_1 + x_2 + 3x_3 = 0$

$x_1 - x_2 + x_3 = 0$

$3x_1 + x_2 - 5x_1 = 0.$

case 1.

$$\frac{x_1}{\begin{vmatrix} 1 & 3 \\ 7 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 3 & 3 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 3 & 1 \\ 1 & 7 \end{vmatrix}}$$

$$\frac{x_1}{-20} = \frac{-x_2}{0} = \frac{x_3}{20}.$$

$(x_1, x_2, x_3)^T = (-2, 0, 2)^T$

case 2.

$$\frac{x_1}{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & 3 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & 1 \\ 1 & 2 \end{vmatrix}}$$

$$\frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{|5|}$$

$(x_1, x_2, x_3)^T = (-1, 1, 1)^T$

case 3

$$\frac{x_1}{\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -5 & 3 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -5 & 1 \\ 1 & -1 \end{vmatrix}}$$

$$\frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4}$$

$(x_1, x_2, x_3)^T = (1, 2, 1)^T$

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* Diagonialisation of a matrix

Work rule

Step1 :- Given a square martix A, we find the Eigen value and Eigen vector.

Step2 :- we find a modal martix P which is a matrix of Eigen Vector.

Step 3. We find $P^{-1} = \frac{adj(P)}{|P|} =$

Step4 :- We compute $d = P^{-1}AP =$ we get the required diagonalisable matrix

Note The diagonal matrix D is also called as spectral martix.

1. Diagonalise $A = \begin{bmatrix} -3 & 8 \\ -2 & 7 \end{bmatrix}$

repeat the process from the problem no.1 of eigen values of eigen vectors.

Eigen values :

$\lambda = 5$ and -1

Eigen vectors :

$X_1 = (1,1)^T$

$X_2 = (4,1)^T$

Model matrix is $P = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$

$P^{-1} = \frac{adj(P)}{|P|} = \begin{bmatrix} -1/3 & 4/3 \\ 1/3 & -1/3 \end{bmatrix}$

D $P^{-1}AP = \begin{bmatrix} -1/3 & 4/3 \\ 1/3 & -1/3 \end{bmatrix} \begin{bmatrix} -3 & 8 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$

$D = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$

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2. Reduce the matrix into diagonal form.

$$A = \begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}$$

Eigen values :- $\lambda = 2, 1$

Eigen vectors

$X_1 = (1, 1)^T$

$X_2 = (3/2, 1)^T$

Modal matrix is $P = \begin{bmatrix} 1 & 3/2 \\ 1 & 1 \end{bmatrix}$

$$P^{-1} = \frac{adj(P)}{|P|} = \begin{bmatrix} -2 & 3 \\ 2 & -2 \end{bmatrix}$$

$$D = P^{-1}AP = \begin{bmatrix} -2 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3/2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

3. Find the modal and spectral matix of

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Eigen values

$\lambda = 1, 2, 3$

Eigen vectrons :-

$x_1 = (-1, 0, 1)^T$

$x_2 = (0, 1, 0)^T$

$x_3 = (1, 0, 1)^T$

Model matrix is $P = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

The spectral matrix is $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

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* Quadratic forms:-

A homogenous equation in any number of variables of 2° degree is called as a quadratic forms

In general, $a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$ represents a quadratic form in two variables similarly a quadratic form in three variables can be return as.

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{23}x_2x_3 + 2a_{31}x_3x_1$$

* Matrix of Quadratic form

(a) for two variables.

$$A = \begin{bmatrix} \text{coeffic of } x_1^2 & \frac{1}{2}\text{ coff } x_1x_2 \\ \frac{1}{2}\text{ coeff of } x_1x_2 & \text{coffic of } x_2^2 \end{bmatrix}$$

(b) for three variables

$$A = \begin{bmatrix} \text{coeff of } x_1^2 & \frac{1}{2}\text{coff } x_1x_2 & \frac{1}{2}\text{ coff } x_1x_3 \\ \frac{1}{2}\text{coeff of } x_1x_2 & \text{coff of } x_2^2 & \frac{1}{2}\text{ coff } x_2x_3 \\ \frac{1}{2}\text{coff } x_1x_3 & \frac{1}{2}\text{ coff } x_2x_3 & \text{coff } x_3^2 \end{bmatrix}$$

* Reducing of Quadractic form x cannonical form. or Sum of squares

Step1 Given QF, we write its matrices

Step2. We find Eigen values of the matrix

Step3. If $\lambda_1, \lambda_2, \lambda_3$ are Eigen the values of matrix then the required cannonical form or sum of square is $\lambda_1 y^2 + \lambda_2 y_2^2 + \lambda_3 y_3^3$

* Rank, Index, signature and nature of QF

1. Rank = No. of non zero eigen values.
2. Index = No. of positive Eigen values.
3. Signature = No. of positive Eigen values − no. of negative Eigen value
4. nature of QF = (i) if all eigen values are positive ⇒ "+ve definite"
   (ii) if all eigen values are negative ⇒ "−ve definite"
   (iii) if any one eigen value is zero and all other eigen value are positive ⇒ "positive semi definite"
   (iv) if any one eigen value is zero and all other eigen value are negative ⇒ "negative semi definite"
   (v) If few eigen value are positive and eigen value negative ⇒ "in definite"

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* Reduce the Quadratic form to. $2x_1^2+2x_2^2+2x_3^2+2x_1x_3$ into cannonical form. Hence find its rank, index, signature and nature.

1. $2x_1^2+2x_2^2+2x_3^2+2x_1x_3$

The matrix of QF is

$$A\begin{bmatrix}2 & 0 & 1\\ 0 & 2 & 0\\ 1 & 0 & 2\end{bmatrix}$$

from problem no (3) of Eigen values. and Eigen vectors, we have.

$(\lambda_1,\lambda_2,\lambda_3)=(1,2,3)$

The cannonical form (or) $=y_1^2+2y_2^2+3y_3^2$. sum of square.

(i) Rank = 3.

(ii) Index = 3

(iii) signature = 3−0 = 3.

(iv) Nature of QF = Positive definite.

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2. Reduce the Quadratic form $8x^2 + 7y^2 + 3z^2 - 12xy - 8yz + 4xz = 0$. sum of squares using orthogonal tranformation method. find its rank, index, signature, nature.

The matrix of QF is.

$$A = \begin{bmatrix} 8 & -6 & +2 \\ -6 & 7 & -4 \\ +2 & -4 & 3 \end{bmatrix}$$

from problem on ④ of Eigen values an Eigen vectors, we have

$(\lambda_1, \lambda_2, \lambda_3) = (0, 3, 15)$

$3y_2^2 + 15y_3^3 = 0$

Rank = 2.

Index = 2

signature = 2-1 = 1

nature = positive semi definite.

3. Reduce the Quadratic form. $x_1^2 + 5x_2^2 + x_3^2 + 2x_1x_2 + 6x_1x_3 + 2x_2x_3$. into cannonical form and find rank, index signature nature.

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

from problem on ③ of Eigen values on Eigen vectors we have

$(\lambda_1, \lambda_2, \lambda_3) = (-2, 3, 6)$

$-2y_1^2 + 3y_2^2 + 6y_3^3 = 0$

Rank = 3

Index = 2

Signature = 2-1 = 1

nature = indefinite.